DOCUMENT RESUME

ED 452 032 SE 064 494

| | |
|---|---|
| AUTHOR | Nakahara, Tadao, Ed.; Koyama, Masataka, Ed. |
| TITLE | Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME) (24th, Hiroshima, Japan, July 23-27, 2000), Volume 2. |
| INSTITUTION | International Group for the Psychology of Mathematics Education. |
| ISSN | ISSN-0771-100X |
| PUB DATE | 2000-00-00 |
| NOTE | 320p.; For volumes 1-4, see SE 064 993-996. |
| PUB TYPE | Collected Works - Proceedings (021) |
| EDRS PRICE | MF01/PC13 Plus Postage. |
| DESCRIPTORS | Algebra; Cultural Influences; *Educational Assessment; Elementary Secondary Education; Functions (Mathematics); Geometric Concepts; Geometry; Graphs; *Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; *Problem Solving; *Statistics; Teaching Methods; Thinking Skills |

ABSTRACT

The second volume of the 24th annual conference of the International Group for the Psychology of Mathematics Education contains full research report papers. Papers include: (1) "What you see is what you get: The influence of visualization on the perception of data structures" (Dan Aharoni); (2) "Exploring the transparency of graphs and graphing" (Janet Ainley); (3) "Describing primary mathematics lessons observed in the Leverhulme Numeracy Research Programme: A qualitative framework" (Mike Askew, Margaret Brown, Hazel Denvir, and Valerie Rhodes); (4) "An analysis of bracket expansion errors" (Paul Ayres); (5) "Knowing the sample space or not: The effects on decision making" (Paul Ayres and Jenni Way); (6) "The development of mathematics education based on ethnomathematics (2): Analysis of Universal Activities in terms of verbs" (Takuya Baba and Hideki Iwasaki); (7) "Maths as social and explanations for 'underachievement' in numeracy" (David A. Baker and B.V. Street); (8) "Year 6 students' idiosyncratic notions of unitising, reunitising, and regrouping decimal number places" (Annette R. Baturo and Tom J. Cooper); (9) "Factors influencing teachers' endorsement of the core mathematics course of an integrated learning system" (Annette R. Baturo, Tom J. Cooper, Gillian C. Kidman, and Campbell J. McRobbie); (10) "Students' conceptions of the integral" (Jan Bezuidenhout and Alwyn Olivier); (11) "The use of mental imagery in mental calculation" (Chris Bills and Eddie Gray); (12) "Readiness for algebra" (Gillian M. Boulton-Lewis, Tom J. Cooper, B. Atweh, H. Pillay, and L. Wilss); (13) "Students' knowledge of length units: Do they know more than rules about rulers?" (Philippa Bragg and Lynne Outhred); (14) "Becoming more aware: Psychoanalytic insights concerning fear and relationship in the mathematics classroom" (Chris Breen); (15) "Same/different: A 'natural' way of learning mathematics" (Laurinda Brown and Alf Coles); (16) "The effect of some classroom factors on grade 3 pupil gains in the Leverhulme Numeracy Research Programme" (Margaret Brown, Hazel Denvir, Valerie Rhodes, Mike Askew, Dylan Wiliam, and Esther Ranson); (17) "'Automatism' in finding a 'solution' among junior high school students: A comparative study" (Gildo Luis Bulafo) (18) "A study of the mathematical behaviors of mathematicians: The role of metacognition and mathematical intimacy in solving problems" (Marilyn P. Carlson); (19) "Bringing out the

Reproductions supplied by EDRS are the best that can be made
from the original document.

algebraic character of arithmetic: Instantiating variables in addition and subtraction" (David Carraher, Barbara M. Brizuela, and Analucia D. Schliemann) (20) "The game of social interactions in statistics learning and in cognitive development" (Carolina Carvalho and Magarida Cesar); (21) "Student misconceptions in interpreting basic graphic calculator displays" (Michael Cavanagh and Michael Mitchelmore); (22) "Step skipping during the solution of partitive quotient fraction problems" (Kathy Charles and Rod Nason); (23) "Making, having and compressing formal mathematical concepts" (Erh-Tsung Chin and David Tall); (24) "Mental projections in mathematical problem solving: Abductive inference and schemes of action in the evolution of mathematical knowledge" (Victor V. Cifarelli); (25) "Solving equations and inequations: Operational invariants and methods constructed by students" (Anibal Cortes and Nathalie Pfaff); (26) "The flow of thought across the Zone of Proximal Development between elementary algebra and intermediate English as a second language" (Bronisuave Czarnocha and Vrunda Prabhu); (27) "A didactic sequence for the introduction of algebraic activity in early elementary school" (Jorge Tarcisio Da Rocha Falcao, Anna Paula Lima Brito, Claudia Roberta De Araujo, Monica Maria Lessa Lins, and Monica Oliveira Osorio); (28) "Towards a definition of attitude: The relationship between the affective and the cognitive in pre-university students" (Katrina Daskalogianni and Adrian Simpson); (29) "A memory-based model for aspects of mathematics teaching" (Gary Davis, David Hill, and Nigel Smith); (30) "Involving pupils in an authentic context: Does it help them to overcome the 'illusion of linearity'?" (Dirk De Bock, Lieven Verschaffel, Dirk Janssens, and Karen Claes); (31) "The difficulties students experience in generating diagrams for novel problems" (Carmel M. Diezmann); (32) "Abcut argumentation and conceptualization" (Nadia Douek and Ezio Scali); (33) "Gesture and oral computation as resources in the early learning of mathematics" (Jan Draisma); (34) "Students' statistical reasoning during a data modeling program" (Lyn D. English, Kathy L. Charles, and Donald H. Cudmore); (35) "The 'mathematics as a gendered domain' scale" (Helen J. Forgasz and Gilah C. Leder); (36) "Investigating function from a social representation perspective" (Janete Bolite Frant, Monica Rabello de Castro, and Flavio Lima); (37) "Definition as a teaching object: A preliminary study" (Fulvia Furinghetti and Domingo Paola); (38) "When a learning situation becomes a problematic learning situation: The case of diagonals in the quadrangle" (Hagar Gal and Liora Linchevski) and (39) "From traditional blackboards to interactive whiteboards: A pilot study to inform system design" (Christian Greiffenhagen). (ASK)

Reproductions supplied by EDRS are the best that can be made
from the original document.

ED 452 032

# Proceedings

of the

## 24th Conference

of the

## International Group for the Psychology of Mathematics Education

PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL HAS BEEN GRANTED BY

J Van Dormolen

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

1

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☒ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

- Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

PME
24
2000
Hiroshima Japan

*July 23-27, 2000*

SE064494

**Editors: Tadao Nakahara**
**and Masataka Koyama**
**Hiroshima University**

**Volume 2**

BEST COPY AVAILABLE

# Proceedings

of the

## 24th Conference

of the

## International Group for the Psychology of Mathematics Education

PME
24
2000
Hiroshima Japan

*July 23-27, 2000*

**Editors: Tadao Nakahara**
**and Masataka Koyama**
**Hiroshima University**

**Volume 2**

# *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*

Volume 2

**Editors:**
Tadao Nakahara
Masataka Koyama

Department of Mathematics Education
Hiroshima University
1-1-2, Kagamiyama, Higashi-Hiroshima 739-8523
Japan

Fax: +81-824-22-7076
E-mail: nakahar@hiroshima-u.ac.jp
E-mail: mkoyama@hiroshima-u.ac.jp

Copyright © 2000 left to the Authors

*All rights reserved*

ISSN 0771-100X

**Logo:**
The logo of the PME 24 Conference has been designed by Ayaka Kunikiyo.

Printed by the Nishiki Print Co., Ltd.
Hiroshima, Japan

# VOLUME 2

Table of contents

**Research Reports**

Aharoni, Dan — 2-1
*What you see is what you get: The influence of visualization on the perception of data structures*

Ainley, Janet — 2-9
*Exploring the transparency of graphs and graphing*

Askew, Mike; Brown, Margaret; Denvir, Hazel & Rhodes, Valerie — 2-17
*Describing primary mathematics lessons observed in the Leverhulme Numeracy Research Programme: A qualitative framework*

Ayres, Paul — 2-25
*An analysis of bracket expansion errors*

Ayres, Paul & Way, Jenni — 2-33
*Knowing the sample space or not: The effects on decision making*

Baba, Takuya & Iwasaki, Hideki — 2-41
*The development of mathematics education based on ethnomathematics (2): Analysis of Universal Activities in terms of verbs*

Baker, David A. & Street, B.V. — 2-49
*Maths as social and explanations for 'underachievement' in numeracy*

Baturo, Annette R. & Cooper, Tom J. — 2-57
*Year 6 students' idiosyncratic notions of unitising, reunitising, and regrouping decimal number places*

Baturo, Annette R.; Cooper, Tom J.; Kidman, Gillian C. & McRobbie, Campbell J. — 2-65
*Factors influencing teachers' endorsement of the core mathematics course of an integrated learning system*

Bezuidenhout, Jan & Olivier, Alwyn 2-73
*Students' conceptions of the integral*

Bills, Chris & Gray, Eddie 2-81
*The use of mental imagery in mental calculation*

Boulton-Lewis, Gillian M.; Cooper, Tom J.; Atweh, B.; Pillay, H. & Wilss, L. 2-89
*Readiness for algebra*

Bragg, Philippa & Outhred, Lynne 2-97
*Students' knowledge of length units: Do they know more than rules about rulers?*

Breen, Chris 2-105
*Becoming more aware: Psychoanalytic insights concerning fear and relationship in the mathematics classroom*

Brown, Laurinda & Coles, Alf 2-113
*Same/different: A 'natural' way of learning mathematics*

Brown,Margaret; Denvir, Hazel; Rhodes, Valerie; Askew, Mike; Wiliam, Dylan & Ranson, Esther 2-121
*The effect of some classroom factors on grade 3 pupil gains in the Leverhulme Numeracy Research Programme*

Bulafo, Gildo Luís 2-129
*"Automatism" in finding a "solution" among junior high school students: A comparative study*

Carlson, Marilyn P. 2-137
*A study of the mathematical behaviors of mathematicians: The role of metacognition and mathematical intimacy in solving problems*

Carraher, David; Brizuela, Bárbara M. & Schliemann, Analúcia D. 2-145
*Bringing out the algebraic character of arithmetic: Instantiating variables in addition and subtraction*

Carvalho, Carolina & César, Margarida 2-153
*The game of social interactions in statistics learning and in cognitive development*

Cavanagh, Michael & Mitchelmore, Michael 2-161
*Student misconceptions in interpreting basic graphic calculator displays*

Charles, Kathy & Nason, Rod 2-169
*Step skipping during the solution of partitive quotient fraction problems*

Chin, Erh-Tsung & Tall, David 2-177
*Making, having and compressing formal mathematical concepts*

Cifarelli, Victor V. 2-185
*Mental projections in mathematical problem solving: Abductive inference and schemes of action in the evolution of mathematical knowledge*

Cortes, Anibal & Pfaff, Nathalie 2-193
*Solving equations and inequations: Operational invariants and methods constructed by students*

Czarnocha, Bronisuave & Prabhu, Vrunda 2-201
*The flow of thought across the Zone of Proximal Development between elementary algebra and intermediate English as a second language*

Da Rocha Falcão, Jorge Tarcísio; Brito Lima,Anna Paula; De Araújo,Cláudia Roberta; Lins Lessa, Mônica Maria & Osório, Mônica Oliveira 2-209
*A didactic sequence for the introduction of algebraic activity in early elementary school*

Daskalogianni, Katrina & Simpson, Adrian 2-217
*Towards a definition of attitude: The relationship between the affective and the cognitive in pre-university students*

Davis, Gary; Hill, David & Smith, Nigel 2-225
*A memory-based model for aspects of mathematics teaching*

De Bock, Dirk; Verschaffel, Lieven; Janssens, Dirk & Claes, Karen 2-233
*Involving pupils in an authentic context: Does it help them to overcome the "illusion of linearity"?*

Diezmann, Carmel M. 2-241
*The difficulties students experience in generating diagrams for novel problems*

Douek, Nadia & Scali, Ezio 2-249
*About argumentation and conceptualisation*

Draisma, Jan 2-257
*Gesture and oral computation as resources in the early learning of mathematics*

English, Lyn D.; Charles, Kathy L. & Cudmore, Donald H. 2-265
*Students' statistical reasoning during a data modelling program*

Forgasz, Helen J. & Leder, Gilah C. 2-273
*The 'mathematics as a gendered domain' scale*

Frant, Janete Bolite; Rabello de Castro, Monica & Lima, Flavio 2-281
*Investigating function from a social representation perspective*

Furinghetti, Fulvia & Paola, Domingo 2-289
*Definition as a teaching object: A preliminary study*

Gal, Hagar & Linchevski, Liora 2-297
*When a learning situation becomes a problematic learning situation: The case of diagonals in the quadrangle*

Greiffenhagen, Christian 2-305
*From traditional blackboards to interactive whiteboards: A pilot study to inform system design*

8

# RESEARCH REPORTS

# WHAT YOU SEE IS WHAT YOU GET — THE INFLUENCE OF VISUALIZATION ON THE PERCEPTION OF DATA STRUCTURES

Dan Aharoni

Department of Education in Science and Technology
The Technion, Israel Institute of Technology

IBM Research Lab in Haifa, Israel

Email: aharoni@netvision.net.il

## Abstract

*The role of visual representation of mathematical concepts has been investigated abundantly in the research of mathematics education. This paper presents evidence indicating the similar role that visual representations plays in thinking process related to Data Structures, one of Computer Science's central topics. The paper shows how visual representations influence students' perception of the overall properties of data structures. It also suggests that people use visual representations to reduce the level of abstraction of the data structures. Some implications for teaching of Data Structures are also presented.*

## 1. Introduction

The role of *mental visualization* or *mental imagery* in the process of concept formation has long since been recognized. Mental visualization generates one of the knowledge representations in the mind — *visual representation* (VR) (e.g. Reed, 1996, pp. 175–218).

Historically, scientists have often made use of VR. Albert Einstein's words, for example, are quoted by the great mathematician Hadamrad (1945/1996):

> The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be "voluntarily" reproduced and combined. (p. 142)

Hadamard also says about himself — and about Schopenhauer:

> [...] I fully agree with Schopenhauer when he writes, "Thoughts die the moment they are embodied by words." (p. 75)

Since mental visualization is such an important factor in thinking processes in general, it is no wonder that, in mathematics education in particular, there is a vast amount of research and theoretical background concerning the role of visualization. Some of these theories are mentioned in this paper.

What about Computer Science (CS) education? CS is mathematical in nature, especially in topics such as *automata theory*, *graph theory*, etc. Nevertheless, research in CS Education (CSE), as compared to that in math education, is still in its infancy; it is almost impossible to find any research on cognitive processes occurring while dealing with CS, let alone theories about visualization.

This paper reports one finding of a research in the domain of *Psychology of Computer Science Education* (The name *Psychology of Computer Science Education* was suggested in Aharoni, 2000). The research, which has recently been completed, investigated undergraduate students' perception of data structures (DSs), one of the central topics in CS. It has already been argued (Aharoni & Leron, 1997) that theories from research on mathematical thinking can be extended to CSE. This paper adds to the generalization of theories about the role of visualization in mathematics, by extending them to DSs.

## 2. A Glance at the Research Design

The research presented in this paper used qualitative methods. It was a case study of nine CS majors taking a course in DS. Semi-structured interviews were used as the main data collection tool. The interview questions covered the following topics: DSs in general, arrays, stacks, queues, linked lists, and the construction of a DS to fit the requirements of a given problem. The questions covered declarative formulations (e.g. "What is an array?"), operative formulations (e.g. "What is required from a DS in order to be called 'an array'?"), operations on DS (e.g. "How can a circle in a linked list be found?"), and more general questions for probing into the student's thinking (e.g. "Is 'variable' a DS?"). In addition to the interviews, classes dealing with data structures were observed and documented. Occasional questioning[1] of computer scientists was also done.

Methods and theoretical frameworks from research on mathematical thinking were used for data analysis.

## 3. The Influence of VR on the Perception of DS

One outstanding behavior of people dealing with DSs in general, and the interviewees in this research in particular, is the extensive use of pictures of DSs; namely, the use of visual representations of data structures. On many occasions where people deal with some DS they sketch it on a paper and use the sketch to track the operations on that DS. Moreover, even when paper is not used, mental images of that DS are used, as explained below.

Edna, one of the students interviewed, was asked to sort the DSs she knew, according to any sorting criterion of her choice. She chose a criterion she defined as "the level of their internal organization". When she was asked to explain, she said (emphasis added):

---

[1] An *occasional questioning* is a questioning which isn't pre-prepared; instead, it is carried out when some occasion occurs, where the researcher identifies an opportunity to question somebody in order to get some more data for the research. This, of course, cannot be the **main** data collection tool, but it may add to data collected in some other, rigor, way.

> A linked list[2], a queue[3], a stack[4], a bi-queue[5] and a tree, they have like some shape that is pictured when talking about them. For example, a linked list is squares one after the other, and a queue and a stack [are] like more or less such a long thing, and a tree, like, circles connected, a lot of circles and such, and a set is something foggy.

The mental images Edna held for the different DSs influenced her thinking. They influenced "the level of internal organization" she saw in each DS.

In another case, Edna suggested a solution for storing information on items in some factory. Since two different types of information per item had to be stored, she suggested working with two linear arrays[6]. She commented that this was not an efficient solution (In the interview segments, "I" denotes "Interviewer"):

> I: You say that it is not efficient; compared to what?
>
> Edna: For example a linked list, a few fields can be built and then a field can simply be added.
>
> I: And two parallel arrays are less efficient in such a case.
>
> Edna: It seems to me.
>
> I: Why?
>
> Edna: Again, it is the picture, because two arrays seem like two tables. A linked list will nevertheless remain a linked list, it only will have an added field, like, visually speaking.

Edna's considerations rely on the visual representations of DSs. She says that the linked list will remain a linked list even if a field is added to every element in the list, since the picture doesn't change much. This is as opposed to working with two arrays, which, as she sees it, turns the arrays into a new entity, namely "two tables", a larger DS, and as such — less efficient. The picture of the linked list remains almost the same if a new field is added to each element, but the picture of two arrays "together" looks quite different from that of a sole array. It is important to note that Edna's perception led her to wrongly evaluate her solution's efficiency; the two arrays solution, in fact, uses less computer storage than the linked list solution.

---

[2] A *linked list* is a collection of elements, where each element holds information indicating which is the next element/s.

[3] A *queue* is a data structure of the FIFO (First In First Out) kind; it behaves like a queue for a bus: a new element is inserted at the queue's end, and an element may be drawn from the queue's head.

[4] A *stack* is a data structure of the LIFO (Last In First Out) kind; it behaves like a vertical stack of books on a library desk: the books are added ("PUSHed") onto the stack one after the other, and when one draws ("POPs") books from the stack, the first to be drawn is the last to have been entered.

[5] A *bi-queue* is like a queue, but elements may be inserted and drawn from both sides.

[6] A *Linear array* is much like a one-dimensional vector in mathematics, namely a collection of elements where each element has an index. Two operations may be done on a linear array: Inserting an element into a specific index, and getting the value of the element which has a specific index.

In another question, Roni was asked to divide the different DSs to groups. Roni included the two DSs *set* and *heap*[7] in the same group. He was asked to explain his considerations.

Roni: Eh, they simply are pictured the same. I simply say that a set is so and so [he draws the set in Figure 1 below], a heap is some such pile [draws the heap] it means, a heap has a fixed order, I can say, like in an array [...] and a set is like [...] such a potato puree.

I: [Laughing] Like a potato puree! Wow, that is a great idea: "A set is like a potato puree"!

Roni: [Continues to divide the DSs to groups, and now deals with the DS "table"] Now a table is not, it is not pictured in my mind to anything. I mean everything here is how you see it, how they are seen, it is more interesting than how they are implemented.

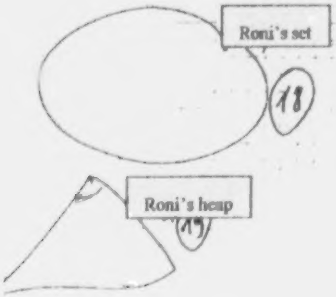


**Figure 1. Roni's perception of the "heap" and "set" data-structures**

Similar to Edna, Roni used VRs to relate some DSs to each other. He too, talks about "how they are seen" as opposed to "how they are implemented". Without saying it explicitly, in Roni's thinking process, as in Edna's, the VRs represent the *abstract data structures,* where "an abstract DS" means "the DS including its properties and operations, without taking into account its implementation" (Leron, 1987).

Another interviewee, Hadas, divided the DSs according to the number of dimensions they have. When he arrived at the DS *tree*, the interview proceeded as follows:

I: OK. Now, how many dimensions do trees have?

Hadas: [...] it is 2-dimensional, I mean you don't go inside, even from the picture

---

[7] A *heap* is a DS which holds values that can be inserted and drawn, and the largest value can always be found in complexity O(1) (i.e. the number of operations needed to find it is constant; it doesn't depend on the value itself).

> point of view. Even a 2-dimensional array, what does it mean? It means that on the page I can put the whole list. The depth is not needed. It is 2-dimensional.

Hadas explained the number of dimensions of the DSs according to their picture "on a page". Here too, in order to consider the DSs' properties, the visual representations of the DSs were used.

We come now to the last example regarding the influence of VR on the perception of a DS's properties. Here we consider the question "Which DS can be empty and which cannot?" Edna dealt with this question, and explained that a *linked list* which has a pointer to its first element can be empty, but a *tree*, or a linked list without a pointer to its first element, cannot. Edna explained (bold letters: Edna's own emphasis):

> Usually when talking about a tree you, like, mean that there is something, at least a root. [...] And if there is nothing then we say there is no tree. [...] Now, a linked list, as it is pictured, it is some element, it has a pointer to another element. It **is possible** to make a linked list that will have some element, like, empty, which [...] is [only] a pointer to the first element. And then if the list is empty, then simply [it is empty and that's all]. But if **we don't have** such an element then, like, there is no empty list. [...] Because we even don't have something to refer to as a list. It is pictured this way.

When looking at Figure 2 below, it is no surprise that Edna perceives some DSs as capable of being empty, and others as not. We can clearly see how VRs influence her perception, and that of other interviewees. The *tree* and the *linked list with no pointer to its first element* simply vanish from the picture when they are empty, while the other type of linked list doesn't. Thus, the first two DSs are perceived as not capable of being empty, while the last is perceived as capable of being empty.

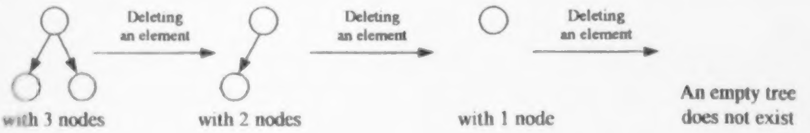
It is worthwhile noting that there is no reason whatsoever why an abstract DS shouldn't be capable of being empty. Moreover, if the abstract DS is capable of being empty, then the concrete DS, i.e. the DS's implementation, must also provide us with this capability. But as we saw, the visual representations of the DSs led the interviewees to the perception that some DS may be empty, while others may not.

The research findings include broader amounts of data and results concerning visualization of DSs. Due to space limitations they will be elaborated elsewhere.

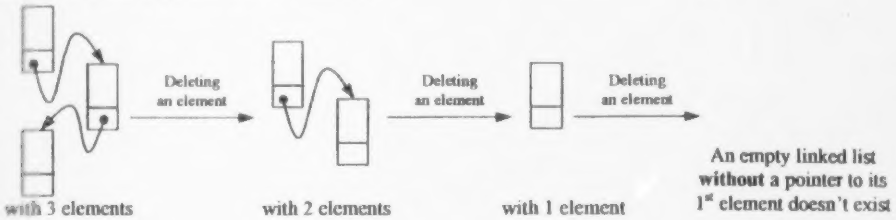
### 4. Discussion

As mentioned previously, there is a vast amount of research and theories regarding the role of *mental visualization* or *mental imagery* in mathematics education. Kaput (1987), for example, presents a theoretical framework for the role of different kinds of representations in the development of meaning. Kaput notes that one stage in the development of mathematical meaning is gaining the ability to translate between mathematical representations and non-mathematical ones. One of these non-mathematical representations is the *concrete visual representation*, exterior to our minds, which aids in building the mental representation of a concept. People dealing with DS really do use concrete visual representations in their work.

A tree:


Deleting an element
Deleting an element
Deleting an element
with 3 nodes
with 2 nodes
with 1 node
An empty tree does not exist

A linked list **without** a pointer to its 1st element:


Deleting an element
Deleting an element
Deleting an element
with 3 elements
with 2 elements
with 1 element
An empty linked list **without** a pointer to its 1st element doesn't exist

A linked list **with** a pointer to its 1st element:

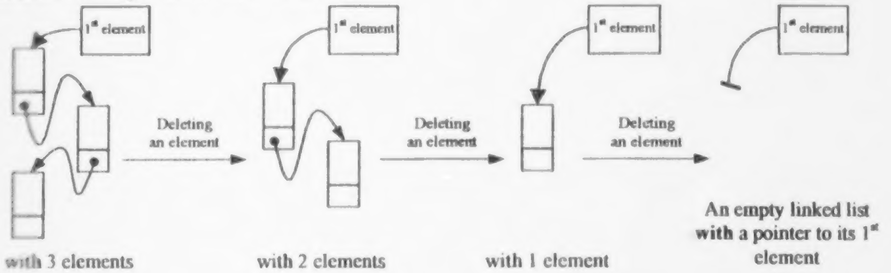

1st element
1st element
1st element
1st element
Deleting an element
Deleting an element
Deleting an element
with 3 elements
with 2 elements
with 1 element
An empty linked list **with** a pointer to its 1st element

**Figure 2. Perception of the emptying process of some data structures**

Dreyfus (1991) talks about the importance of VRs of mathematical concepts, and observes about himself:

> When I think of a vector space, I may "see" arrows (before my mind's eye), and I may be able to think in terms of these arrows when dealing with bases, transformations, etc.

Eisenberg (1991) talks about the importance of VRs when understanding of the concept of function is developed. He argues that in this case VRs are even more important than the analytic framework.

DSs are mathematical in nature, so it was expected that VRs would influence thinking processes that deal with DSs, as they do in mathematics. The research that is reported in this paper indeed provides some evidence that this influence exists.

Why do people dealing with DSs use VRs so extensively? It seems that the VR of the DS at hand reduces the DS's level of abstraction, from the level of a completely abstract idea expressed in a definition, to a more concrete level, in this case — a picture, be it a concrete one (on a page) or a mental one. The picture of the DS usually represents the abstract DS. Hazzan (2000) provides more insight on how students reduce the abstraction level of mathematical concepts.

It was said above that the picture provides concreteness on one hand, yet represents the abstract data type on the other. These two statements seem to contradict each other; however, if we pay attention to the meaning of the terms "abstract" and "concrete", they do not. When we talk about reducing the abstraction level to a *concrete* level, the term "concrete" means "something familiar", as in the sentence "it is too abstract for me; please talk more concretely". The interviewees looked for a representation that would give them the feeling of comfort and familiarity. On the other hand, when we talk about the VR being a representation of the *abstract* DS, we take the meaning of the term "abstract" as the one used in Computer Science: The picture of the DS doesn't include the details of that DS's implementation, only the details of that DS's **properties and organization**. A support to the ideas stated in this paragraph may be found in Sfard (1991):

> Mental images, being compact and integrative, seem to support the structural conception. [...] mental images can be manipulated almost like real objects. (p. 6)

Being able to manipulate the DS at hand almost like a real object is the very heart of the reduction of that DS's abstraction level. At the same time, when referring to the DS through a specific VR, that VR's properties are projected onto the DS. This aids in the conceptualization of that DS, but we also saw that it may result in misconceptions of some of that DS's properties.

## *5. Conclusion and Some Implications for DSs Teaching*

This paper presented two important roles that VRs of DSs play in thinking processes: First, we saw how they influence perception of the overall nature of a DS. Second, it was argued that the VRs are used to reduce the level of abstraction.

As noted, novices as well as experts widely use VRs of DSs. VRs that are acquired during the first stages of the learning process are likely to remain later on. Therefore, teachers should be well aware of the phenomena related to VRs, and use VRs appropriately. In short, the following three main ideas are worth paying attention to:

- Use VRs when explaining DS issues and when solving problems. They aid in reducing the abstraction level and creating familiarity with the DS at hand.
- Prepare adequate VRs which reflect the properties of the DSs, but avoid noisy information in the pictures. This is a very general idea; specific details concerning each and every DS are yet to be figured out.
- Avoid pitfalls in the VR. The picture is not the DS itself; even the best-designed picture may still contain some pitfalls. In such a case, deal with these pitfalls

specifically, by spelling them out to the learners and by confronting these pitfalls with appropriate problems given to the learners.

So remember, **what you see is what you get**; let's make our students see the appropriate picture!

### Acknowledgements

I would like to thank Prof. Uri Leron from the Department of Education in Technology and Science, at the Technion, Israel Institute of Technology, who supervised this research and enlightened my way through it.

### References

Aharoni, D. (2000). Cogito, Ergo Sum! Cognitive Processes of Students Dealing with Data Structures. Proceedings of the Thirty-first SIGCSE Technical Symposium on Computer Science Education (SIGCSE 2000).

Aharoni, D., & Leron, U. (1997). Abstraction is hard in computer-science too. In E. Pehkonen (Ed.), *Proceedings of the 21st conference of the International Group for the Psychology of Mathematics Education (PME21): Vol. 2* (pp. 9–16). Lahti, Finland: University of Helsinki.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (chap. 2, pp. 25–41). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Eisenberg, T. (1991). Functions and associated learning difficulties. In D. Tall (Ed.). *Advanced mathematical thinking* (chap. 9, pp. 140–152). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Hadamard, J. (1996). *The mathematician's mind: The psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press. (Original work published 1945)

Hazzan, O. (2000). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*, 71-90.

Kaput, J.J., (1987). PME 11 Algebra papers: A representational framework. In J.C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the eleventh International Conference for the Psychology of Mathematics Education (PME11): Vol. I* (pp. 345–354). Montréal.

Leron, U. (1987). Abstraction barriers in mathematics and computer science. In J. Hilel (Ed.), *Proceedings of the third International Conference for Logo and Mathematics Education.*

Reed, S.K. (1996). *COGNITION: Theory and applications.* Pacific Grove: Brooks/Cole Publishing Company.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

# EXPLORING THE TRANSPARENCY OF GRAPHS AND GRAPHING

Janet Ainley
Mathematics Education Research Centre, Institute of Education
University of Warwick, Coventry, U. K.

*The increasingly widespread use of graphs in advertising and the news media seems to be based on an assumption, widely challenged by research evidence in mathematics and science education, that graphs are transparent in communicating their meaning. Meira's (1998) view of transparency as emerging through use, rather than being an inherent feature of instructional devices, is used to consider the transparency of graphs as tools, and question traditional curriculum design.*

## Introduction

This paper explores the notion of *transparency* in the context of the use of graphs and graphing in mathematics classrooms. The discussion will make use of Meira's (1998) description of transparency as *an index of access to knowledge and activities* to consider the transparency of graphs as tools, and the ways in which notions of relative transparency may inform decisions about the introduction of graphs in the school curriculum.

I shall also draw on Nemirovsky and Monk's (2000) model of *fusion* in the use of symbols, which they define as, *acting, talking and gesturing without distinguishing between symbols and referents.* Despite the authors' claim that the metaphor of fusion is more appropriate to symbolising than that of transparency, I feel these metaphors may be seen as complementary, with fusion offering a description of how transparency is brought about. Both metaphors will be used here to take account of the dual nature of graphs as both symbols and technical tools.

## Perceptions of graphs

The increasingly widespread use of graphs of many kinds in advertising and the news media for communication and persuasion seems to be based on an assumption that graphs are *transparent* in communicating their meaning. However, this assumption is at odds with the considerable body of research on pupils' difficulties with graphing in mathematics and science education (e.g. Dreyfus & Eisenberg, 1990, Kerslake, 1981, Swatton & Taylor, 1994). The availability of technological tools which can be used to produce graphs is currently provoking re-examination of how the skills of using and interpreting graphs are learnt and taught (e.g. Ainley, 1995, Ben-Zvi & Friedlander, 1996, Phillips, 1997, Pratt, 1995). One aspect of this re-examination is a shift in the perception of the role of graphs from being seen as the product and end point of an activity, to being used as tools within an activity.

Although graphs may primarily be seen as symbolic tools, graphs produced using a computer (in particular with spreadsheets) have significant differences from graphs produced by pencil-and-paper methods which make them much more like technical tools or instructional devices:

- they are dynamic, in the sense that their size and proportions can be altered by dragging the corners of the graph, and the scales shown on the axes may change as the graph is distorted,
- they allow the same data to be quickly and easily displayed in a wide range of graphical forms,
- their appearance can be changed through menus which control the scales on the axes, the orientation, the style of markings and labels, and so on,

These features allow computer graphs to be manipulated by pupils as they explore trends and patterns within data, in ways which may parallel the manipulation of physical instructional devices (Ben-Zvi, 1999). Conceptualising graphs as manipulable tools offers the opportunity for a new perspective on the transparency of graphs. Graphs are unlike many other instructional devices in that they do not exist solely in classrooms. However, the range of types graphs used in the classroom, and the range of contexts to which they relate, will both be much narrower than that of graphs found on a day-to-day basis in the media.

Decisions about the range of graphs used in schools, and the way in which the introduction of different types of graph is structured within school curricula appear to be based upon what Meira (1998) refers to as an *epistemic fidelity* approach to judging the transparency of graphs. In other words, transparency (and hence the relative ‘difficulty’) is conceptualised as being inherent in the graph itself, and progression in the teaching of graphing is based on the use of increasingly complex graphical forms.

The National Numeracy Strategy Framework in the U.K. offers a typical example of such a curriculum: block graphs are used for 5-7 year-olds, bar graphs are introduced for eight year-olds, and continue to be used throughout the primary school in various forms, particularly as frequency graphs, with increasingly complex scales. Line graphs and pie charts are introduced at ages ten and eleven respectively, and scatter graphs do not appear at all within the primary school. (DfEE, 1999). This ‘progression’ is clearly closely tied to the relative difficulty of constructing graphs accurately by hand. Thus the use of scales is a major marker of increased difficulty, and pie charts, which require additional mathematical knowledge in their construction, appear late in the sequence.

However, the appropriateness of this curriculum structure is challenged by research evidence which suggests that when the necessity to draw graphs by hand is removed, particularly in the context of computer graphs, children in the primary age range are able to work with line and scatter graphs successfully (Ainley, 1995, Bryant & Somerville, 1986, Phillips, 1997, Pratt, 1995).

More generally, Meira (1998) challenges the epistemic fidelity construction of transparency which focuses on intrinsic qualities of the devices or displays, and the correspondence between features of the device and a target knowledge domain. In contrast to this, he claims that:

> *the transparency of devices follows from the very process of using them. That is, the transparency of the device emerges anew in every specific context and is created during the activity through specific forms of using the device." (Meira, 1998, p. 138)*

In this paper this conception will be used to explore the emergence of transparency as graphing is shaped in use by pupils, and to identify features which contribute to increased transparency, which may be used to make predictions of pupils' likely trajectories through activities, and thus to inform the design of graphing tasks and a restructuring of the curriculum.

There is room here for only a small number of examples drawn from an extended project (Ainley, Nardi & Pratt, 1998) within which children's responses to, and use of, particular graphs, and their understanding of the role of graphing, varied considerably. In some cases, features of the activity settings made graphs relatively transparent for some children, whilst in other contexts children struggled to make sense of particular graphs and of the role they were intended to play. Examples are offered here illustrating a range of settings and responses, followed by a discussion of significant features affecting the transparency of graphs.

## Intuitive reading of graphs

A number of incidents were observed in which groups of children seemed to respond to and read graphs intuitively: that is, without any direct teaching or overt discussion about features of the graph. For example, a group of six year-olds measured their heights, recorded them on a spreadsheet, and produced a graph (Figure 1). Describing the graph, Kati said, "*The graph is not quite right, because Oliver* [oliva] *is actually the tallest, and Kim and Tom are about the same size*".

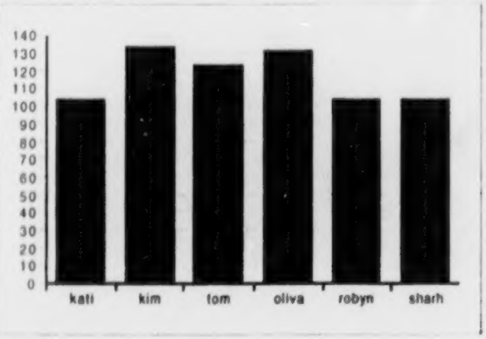


Figure 1. Kati's graph

This graph is unlike those which Kati was likely to have seen previously in school: it has no vertical gridlines, no titles on the axes, it uses continuous bars rather than defined 'blocks', and has a scale on the vertical axis to cope with large numbers.

Kati's reading of the graph took no apparent account of these features. To her the graph seemed completely transparent: she focused directly on the information to be gained from the graph, and related this to the context from which the data was collected. Kati's description of what was wrong demonstrates her fusion between the appearance of the graph and it's human referents: "*the graph is not quite right, because Oliver is actually the tallest.*"

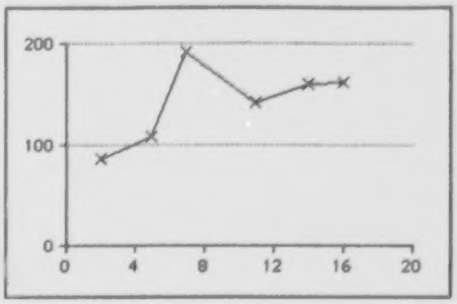


Figure 2. Danny's graph

Another example of intuitive reading comes from the work of eight and nine year-olds who were given data for the heights of imaginary children at different ages, and asked to produce line graphs which they could use to find how tall these children were at other ages. One small incident from this lesson illustrates the way in which pupils imaginatively engaged with the problem, creating what Nemirovsky and Monk (2000) describe as '*a lived-in space in which the absent is made present and ready at hand*'. Two girls produced the graph shown in Figure 2, but explained to their teacher that it could not be right because it looked as though "*Danny has shrunk as he got older.*"

As in the previous example, these children were working with a graph which was unlike those that they had used before in the classroom. They had never used line graphs of any kind: and in addition, this graph has no vertical gridlines, and uses scales on both axes. Again, these children seemed to treat the graphs as transparent. They were able to work with the features of the graph which were important for the task without explicit discussion of how to handle scaled axes, or of the meaning of a line graph.

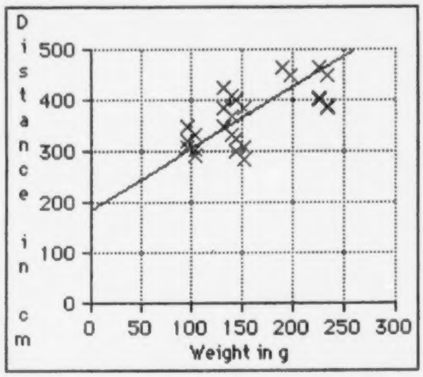
These different classroom examples have two significant features in common which clearly affect the ways which the children were able to use the graphs, and their consequent transparency:

- the data which was being graphed was meaningful for the children from their personal experience,
- the superficial appearance of the graphs (vertical bars and an ascending line) correspond in a straightforward way to the ideas of height and growth which they represent.

We describe this correspondence between the form of the graph and the phenomenon it is used to represent as *metaphoric resonance* to indicate the ways in which descriptions may be used ambiguously to fuse references to the graph and the phenomenon. High metaphoric resonance is likely to increase transparency, and to be associated with fluency in symbolising and intuitive reading of graphs. Where metaphoric resonance is absent, for example in distance-time graphs, difficulties in interpretation caused by the distraction of the superficial appearance are well documented (Kerslake, 1981). Indeed these difficulties seem in part to arise from the *expectation* of a metaphoric resonance which does not exist.

**Learning to work with graphs**

Several tasks were used within the project to focus on developing skills in reading graphs arising from scientific investigations. Children recorded and graphed data from an experiment, and discussion focussed explicitly on making links between the data recorded on the spreadsheet and points shown on the graph. In the following example, a group of nine and ten year-olds were investigating the effects of adding weight to a toy car which they rolled down a slope. A scatter graph was used to compare the distance the car traveled with the weight of the car. An extract from the group's report is shown in Figure 3. Despite their difficulties with spelling, the brief comment indicates that these pupils had made some sense of both the relationship between the variables, and the way this was represented on their graph.



The line showes the trend on the scatergraf.

On the scatergaf it shows you that the hever [*heavier*] the car the ferver [*further*] it goes.

Figure 3. A report on the cars activity

While working on this problem the children manipulated the graph in a number of ways, partly to investigate options offered in the software, and partly in an attempt to display the information more clearly. In particular, they used a square grid (an option offered by the software), chose to start both axes from zero (by using nested menus), and added a line created with drawing tools to emphasise the trend. This sophisticated manipulation of the graph was relatively unusual: more commonly children dragged the graph window to alter its appearance directly in order to create the effect they wanted and to improve the transparency of the image.

As the children collected data, they made graphs several times to see how their latest experiments altered the pattern of results, rather than the graph being produced at the end to illustrate the outcome of the task. This meant that they saw several similar graphs within a short period of time. This was also true in the 'growth graph' activity described earlier, when similar graphs were used for different imaginary children. Within the project, most lessons involved a mixture of group work and whole class discussion, during which groups would show their work to the rest of the class. Thus there were regular opportunities for children to see a number of graphs relating to similar data, as well as a forum for public reading of graphs by pupils and teachers.

**Struggling with the use of graphing**

The examples given so far have featured successful use of graphs. However, observations of incidents in which children responded to graphs in unexpected ways and were less successful in using them, also give valuable insights into the relative transparency of graphs in different contexts.

In a number of incidents we observed that children seemed to be applying aesthetic rather than scientific criteria in their choice of what made a 'good graph'. Several examples of this kind occurred when a group of eleven year-olds were working on a task which involved comparing the quality of life in cities around the world. The children were provided (on disk) with data about various aspects of life in each city (e.g. its population, the murder rate, infant mortality, the provision of basic services and secondary schools), and asked to make comparisons to find the poorest city. They were encouraged to use graphs both as tools to explore the data, and as a medium for communicating their conclusions in a display for the rest of the class. The complexity of this task was increased because each category of data was given in a different measure (e.g. population in millions, infant mortality as a percentage of live births).

Some groups responded by focusing on a single factor, such as living space, as an indictor of wealth or poverty, and exploring different graphs to present comparisons. It was clear that in many cases the main criterion for the choice of a graph was aesthetic, with little concern for the transparency with which the graph conveyed information. The use of colour and graphic effects was particularly popular.

In contrast Jake's group selected a simple horizontal bar graph (Figure 4) to justify their choice of Johannesburg as the poorest city. Jake recognised that "*all the things on the graph were different*" (i.e. used different scales), but said that he felt it gave "*an overall picture of Johannesburg*". In making their presentation, the group showed this graph alongside a similar one for New York (as an example of a rich city).

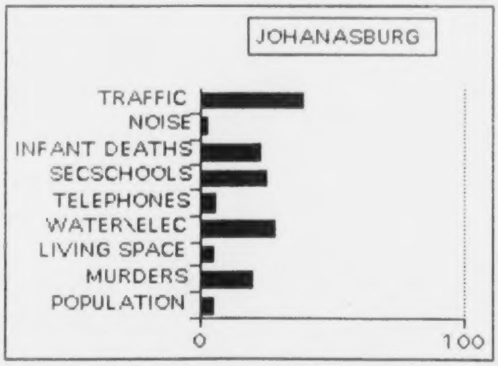


Figure 4. Jake's graph

This group's response shows a relatively sophisticated understanding of the need to make some overall comparison between the cities, rather than focussing on a single feature, but the children seemed unconcerned by the difficulties in interpreting and comparing the graphs they produced. Jake did point out one problem he had noticed: the bar for murders was about three times as long as the one for the population. This provoked an interesting discussion in which some pupils focussed on finding explanations which fitted the appearance of the graph (*this might be all the murders ever committed in the city*), rather than on trying to read the graph accurately, or considering why the appearance of the graph might be inappropriate.

### Features contributing to the transparency of graphs and graphing

The range of children's apparent understanding the nature of graphs and the purposes of graphing shown in this examples may obviously be accounted for in terms of differences between individual children. However, in comparing a wide range of incidents particular features of the pedagogic settings have emerged as significant factors in the relative transparency with which children were able to use graphing.

*The presentation of a complete image* allows children to take a holistic view of the graph, rather than focussing on separate components as the graph is constructed, so that its transparency is blurred. This supports intuitive reading of the graph, particularly in situations where there is high metaphoric resonance. Access to technological tools does not remove the need for the skills and knowledge required to produce graphs by hand, but these may be learned much more easily once children are confident in working with computer graphs.

*The use of a number of similar graphs* sharpens children's discrimination in focussing on similarities and differences. In working with a complex image like a graph, children are often not aware of which features they should attend to, and which can be ignored for particular purposes.

*The ability to manipulate the graph* and change its appearance helps to emphasise features of the data, and supports discrimination between features specific to this data set, and those which are conventions of particular types of graph.

*A familiar and/or meaningful context* which children can imaginatively enter allows children to feel ownership of the data and to make sense of it. The 'quality of life' task is an example of how children were likely to lose sight of the role of graphs and to find difficulty reading them when working in unfamiliar contexts.

*A purposeful task in which the graph is used to solve a problem* provides a structured context for the information contained in the graph, supporting children in making links between the image of the graph and the data. Less successful use of graphing has often been associated in our data with tasks which lack an obvious purpose from the children's perspective. Again the 'quality of life' task provides an example. The teacher saw clear purposes for the activity: giving opportunities to explore the use of different kinds of graphs for comparing sets of data, and leading children to consider the different ways in which wealth or poverty might be quantified. However, the structure of the task was not sufficiently transparent to make either of these purposes clear to the children, resulting in rather unfocussed use of graphing, even when relatively 'simple' types of graph were chosen. This lends support to the conjecture that transparency emerges through the use of graphs, rather than being an inherent quality of the type of representation.

This analysis suggests that it would be appropriate, with the increasing availability of technological tool, to consider a restructuring of the introduction of graphing in the school curriculum, with progression based less on the characteristics of particular types of graphs than on those of the contexts and tasks within which they are used.

Ainley, J. (1995). Re-viewing Graphing: traditional and intuitive approaches. *For the Learning of Mathematics ,15(2)*, 10-16.

Ainley, J., Nardi, E., & Pratt, D. (1998). Graphing as a Computer Mediated Tool. In A. Olivier & K. Newstead (Eds.), *Proceedings of the Twenty Second Annual Conference of the International Group for the Psychology of Mathematics* (Vol. 1, 243-258). Stellenbosch: University of Stellenbosch.

Ben-Zvi, D. (1999). Constructing an understanding of data graphs. In O. Zaslavsky (Ed.) *Proceedings of the Twenty Third Annual Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 2, 97-104) Haifa: Technion.

Ben-Zvi, D. & Friedlander, A. (1996). Statistical Thinking in a Technological Environment. In J. B. Garfield and G. Burrill (Eds.), *Research in the Role of Technology in Teaching and Learning Statistics: Proceedings of the 1996 IASE Round Table Conference* (45-55). Granada: International Statistics Institute.

Bryant, P. & Somerville, S. (1986). The Spatial Demands of Graphs. *British Journal of Psychology*, 77,187-97

Department for Education and Employment (DfEE) (1999). *The National Numeracy Strategy Framework for Teaching Mathematics from Reception to Year 6.* Cambridge: Cambridge University Press.

Dreyfus, T. & Eisenberg, T. (1990). On difficulties with diagrams: Theoretical issues. In G. Booker, P. Cobb & T. N. De Mendicuti (Eds.)*Proceedings of the 14th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, 27-36) Oaxtepex: PME.

Kerslake, D. (1981). Graphs. In K. M. Hart (Ed.), *Children's Understanding of Mathematics: 11-16* (120-136). London: John Murray.

Meira, L. (1998). Making Sense of Instructional Devices: the emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education, 29(2)* ,121-142

Nemirovsky, R. & Monk, S. (2000) "If you look at it the other way ..." An exploration into the Nature of Symbolizing. In P. Cobb, E. Yackel & K. McClain (Eds.) *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools and instructional design*. Hillsdale NJ: Lawrence Earlbaum Associates

Phillips, R. (1997) Can Juniors Read Graphs? *Journal of Information Technology in Teacher Education, 6 (1)*, 49-58.

Pratt, D. (1995) Young Children's Active and Passive Graphing. *Journal of Computer Assisted Learning, 11*, 157-169.

Swatton, P. & Taylor, R. (1994) Pupil Performance in Graphical Tasks and its Relationship to the Ability to Handle Variables. *British Educational Research Journal, 20*, 227-243.

# DESCRIBING PRIMARY MATHEMATICS LESSONS OBSERVED IN THE LEVERHULME NUMERACY RESEARCH PROGRAMME: A QUALITATIVE FRAMEWORK

Mike Askew, Margaret Brown, Hazel Denvir and Valerie Rhodes
King's College, University of London

*This paper describes a theory based descriptive framework being developed as an analytical instrument for part of the Leverhulme Numeracy Research Programme. Building on Saxe's* (1991)*four parameter model, a framework is being developed for examining mathematics lessons in primary (elementary) classrooms. The framework is required to help us understand English pupils' progress over five years for two cohorts of children. It is anticipated that this framework will help us better understand why the classes of some teachers in the programme made greater gains than other classes on specially designed tests of numeracy.*

## INTRODUCTION

The Leverhulme Numeracy Research Programme (LNRP) is a longitudinal study of the teaching and learning of numeracy *investigating factors leading to low attainment in primary (elementary) numeracy in English schools, and testing out ways of raising attainment.* Two cohorts of children, one starting in Reception (four and five year-olds) and one in Year 4 (eight and nine year-olds), are being tracked through five years of schooling. Pupil data are being collected at several different levels ranging from large scale, twice yearly, assessments on each of the two cohorts (some 1700 children in each cohort) through to detailed case studies of six children from each cohort in five schools. Further details of data collection are given in a parallel paper presented at this conference (Brown, Denvir, Rhodes, Askew, Wiliam, & Ranson, 2000).

The cohort of eight and nine year-olds were assessed on a specially designed test of numeracy in the Autumn term of 1997 and again the following June. (The younger cohort were considered to be too immature to be tested in the first year of the programme but were tested at the beginning and end of the 1998/99 school year). On the basis of individual pupils' gains on the assessment across the two points, mean class gains were calculated for the 73 classes involved. (See Brown et. al. (2000) for details of the design of the assessment and procedures for calculating gains). These mean class gains were matched against a range of data gathered from a questionnaire administered to their teachers.

Whilst few of the teaching variables elicited from the questionnaire had a significant statistical association with class mean gain we, the researchers who observed the mathematics classrooms, felt that we were able, to some extent, to predict which classes would make large or small overall gains. Significant differences between classes, we believed, could be observed in the classrooms and concerned subtler aspects of the teaching and classroom relationships than are readily obtained from questionnaire data. This view is supported by other studies (see, e.g. Wiliam, (1992) also by Galton (1989) who finds that the nature of the relationship between teacher and pupils is crucial.

This paper presents a theory based descriptive framework developed to help us explore qualitative differences in lessons and in particular in understanding interactions in primary mathematics lessons and the potential impact on pupils' learning.

## THEORETICAL BACKGROUND

As Sfard (1998) points out, theories of learning can broadly be divided along the lines of whether they rest upon the metaphor of 'learning as acquisition' or 'learning as participation'. 'Learning as acquisition' theories can be regarded broadly as mentalist in their orientation, with the emphasis on the individual building up cognitive structures (Alexander, 1991; Baroody & Ginsburg, 1990; Carpenter, Moser, & Romberg, 1982; Kieran, 1990; Peterson, Swing, Stark, & Waas, 1984). In contrast 'learning as participation' theories attend to the socio-cultural contexts within which learners can take part (Brown, Collins, & Duguid, 1989; Lave & Wenger, 1991; Rogoff, 1990).

While some writers argue for the need for a paradigm shift away from (or even rejecting) acquisition perspectives in favour of participation, we agree with Sfard in the suggestion that the metaphors are not alternatives but that both are necessary and each provides different insights into the nature of teaching and learning. However, we would argue that participation in some sense precedes acquisition. Pupils' learning can be examined through analysis of their 'participation in sociocultural activities' (Rogoff, 1995) and learning is regarded as occurring through *changes in and transformation of* such participation. Thus the focus of attention within the classroom is the nature and content of the sociocultural activities as determined by the provisions made by the teacher and the interactions of teacher and pupils within the lessons. Analysing classroom practices within a framework of 'transformation of participation' (Rogoff, 1995) provides a discourse that examines the sociocultural activities in which pupils have the opportunity to participate.

## DEVELOPMENT OF THE FRAMEWORK

As a starting point for examining participation in and 'transformation of participation' events, we have adapted Saxe's (1991) four-parameter model for examining emergent goals, adapting his four parameters. Our four main parameters are task, talk, tools and relationships & norms. These parameters were used to create some initial framing questions.

**Tasks**

- Mathematical challenge: To what extent does the lesson content/tasks mathematically challenge all pupils appropriately?
- Integrity & significance: Are the tasks of the lesson developed in ways that draw out mathematical/didactical integrity and significance?
- Engage interest: Do the lesson's tasks engage pupils' interest in the mathematical content?

**Talk**

- Teacher talk: To what extent does the teachers' talk focus on mathematical meanings and understandings as co-constructed and not simply transmitted?
- Teacher-pupil talk: To what extent do teachers and pupils engage in discussion about the mathematics?
- Pupil talk: To what extent does the lesson encourage pupils to talk mathematically and display reasoning and understanding?
- Management of talk: To what extent is the lesson managed by the teacher to encourage engagement of all pupils in talk about mathematics?

**Tools**

- Range of Modes: Do the learning tools cover a range of modes of expression, e.g.: oral, visual, kinaesthetic?
- Types of models: Are the models of mathematics didactically "good"?

**Relationships & norms**

- Community of learners : To what extent do teacher and pupils participate as a community of learners?
- Empathy: To what extent does the teacher display empathy with the pupils' responses to the lesson, both affective and cognitive

By examining our observation notes for the 15 classes with the highest and the 15 with the lowest mean class gains and trying to distinguish commonalities within and differences between these, we went on to develop a taxonomy of events and practices within the classroom which we believe have the potential to be strongly related to learning outcomes. Details are given in tables 1 to 4.

| | | | |
|---|---|---|---|
| **Mathematical challenge** | | | |
| All/nearly all pupils are appropriately challenged mathematically, e.g.<br>• most of pupils, most of the time appear to be doing mathematics which challenges them to think mathematically<br>• pupils have some control over level of difficulty | About half the pupils are appropriately challenged all of lesson/all pupils appropriately challenged for a part of the lesson, e.g.<br>• good differentiation in main part of lesson, plenary/intro not adequately differentiated | Some pupils are doing appropriately challenging work for some of the time. | No/very few Pupil are doing work which is appropriately challenging |
| **Integrity & significance** | | | |
| Lesson tasks have high mathematical integrity and significance, e.g.<br>• pupils are encouraged to develop significant connections either within maths or to the application of ideas.<br>• pupils are encouraged to generalise on the basis of the outcomes of tasks | Lesson tasks have some math'l integrity and significance, e.g.<br>• pupils can describe different ways of checking their answers; where they might use this mathematics. | While lesson is mathematically sound, it is not enacted in such a way as to draw on integrity or significance, e.g.<br>• opportunities for pupils to make appropriate connections between different aspects of mathematics or to appropriate applications are not drawn upon<br>• pupils are not encouraged to generalise from the tasks. | Pupils carry out tasks as routines and appear to have no meanings or connections beyond this. |
| **Engage interest** | | | |
| All/nearly all the pupils are engaged in doing mathematics for nearly all of the lesson | Around half of the pupils are engaged/ most pupils are engaged for around half of the time | A few pupils are engaged/most pupils are engaged for a little of the time in doing mathematics | Virtually no pupils are engaged in doing mathematics |

**Table 1: Aspects of tasks**

| | | | |
|---|---|---|---|
| **Teacher talk** | | | |
| Teacher talk demonstrates high level of attention to developing shared mathematical meanings, e.g.<br>• awareness of distinction between objects or symbols and mental objects.<br>• draws on different models to provide further explanations | Teacher talk demonstrates some attention to developing shared mathematical meanings, e.g.<br>• teacher provides elaborated explanations | Teacher talk demonstrates belief in meaning residing in texts rather than being brought to them by the subject, e.g.<br>• teacher gives reasons and/or meanings but assumes that a single way of explaining will convey meaning | Teacher talk demonstrates belief in mathematical meaning simply needing to be transmitted, e.g.<br>• instructs pupils in procedures.<br>• no reasons given that might help develop relational understanding. |
| **Teacher-pupil talk** | | | |
| Teacher-pupil talk displays high level of features of discussion, e.g.<br>• provides mathematical feedback on pupils explanations.<br>• pupils take initiative to feedback to teacher, seek clarification and ask questions | Teacher-pupil talk displays some features of discussion, e.g.<br>• teacher feedback to pupils on reasons and explanations is praised irrespective of mathematical content.<br>• pupils only communicate their understandings to teacher when asked | Teacher-pupil talk is directed by teacher and does not feature any elements of discussion, e.g.<br>• feedback provided to pupils by teacher limited to whether or not answers are correct<br>• pupils expected to feedback correct answers to teacher. | No teacher-pupil talk beyond closed questions and answers |
| **Pupil talk** | | | |
| Many pupils are frequently encouraged to provide extended accounts of reasons / understandings. | Some pupils are afforded the opportunity to provide extended accounts of reasons and understandings | Opportunities to encourage pupils to display reasoning are not used, e.g.<br>• questions do require pupils to figure out answers but little opportunity for extended explanations. | Pupils do not express their reasons or understandings |

BEST COPY AVAILABLE

| **Management of talk** | | | |
|---|---|---|---|
| Lesson is skilfully managed to encourage maximum participation of all pupils, e.g.<br>• pupils are encouraged to engage with each others' explanations, questioning and seeking clarification<br>• teacher checks that pupils are attending to each others explanations | Lesson is managed to encourage participation of pupils who are selected to talk, but not the engagement of the others. | Opportunities to engage pupils in talk are not used, e.g.<br>• pupils explain reasons and understandings but mainly to the teacher.<br>• pupils do not engage with each others' explanations | Talk is not managed at all effectively |

**Table 2: Aspects of talk**

| **Range of Modes** | | | |
|---|---|---|---|
| The tools provided for the lesson enable most pupils to engage with those most suited to their learning styles, e.g.<br>• range of modes to suit range of learning styles/preferences<br>• pupils encouraged to select and work with tools of their choice. | The tools provided for the lesson enable some pupils to engage with those most suited to their learning styles, e.g.<br>• some range of modes but not all<br>• teacher determines which tools to use. | The tools provided for the lesson only enable pupils to engage infrequently with those most suited to their learning styles, e.g.<br>• some range of modes but not necessarily within one lesson. | No attention to learning styles, e.g.<br>• modes limited to one only (e.g. 'chalk and talk' or only unifix cubes) |
| **Types of models** | | | |
| Models used are appropriate and effective for tasks, e.g.<br>• models are useful in going from "model of" to "model for". | Range of models used but teacher does not draw pupils attention to the limitations of any particular models. | Models provided may lead to unitary view of concepts, e.g.<br>• fractions always as bits of pizza, subtraction as 'take-away' | Models only provide window dressing. |

**Table 3: Aspects of tools**

| | | | |
|---|---|---|---|
| **Community of learners** | | | |
| Teacher explicitly works on developing classroom as a community of learners, e.g.<br>• norms are explicitly communicated to pupils.<br>• teacher provides explicit feedback to pupils on expected norms. | Teacher appears to have expectations of norms that might lead to a community of learners but does not explicitly share these with pupils. | Classroom norms separate teacher and pupil - roles are not seen as mutually dependent, e.g.<br>• I teach - you learn<br>• you discover - I facilitate | Classroom norms based on mathematics as jobs to be done rather than ideas to learn about |
| **Empathy** | | | |
| Teacher displays high level of empathy with all pupils and encourages them to feel good about themselves as learners of mathematics even with this involves struggle. | Teacher displays some empathy with some pupils | Teacher displays awareness of pupils affective and cognitive responses but does not work with these. | Teacher displays little/no empathy with pupils. |

**Table 4: Aspects of relationships and norms**

## FUTURE WORK

We are currently using a detailed grading system based on this framework to see if we can predict from our observations the order of gains in mean class scores during the school year 1998/9, of both cohort 2 in Year 5 and cohort 1 in Year 1. Meanwhile we are also analysing as part of one of the other projects on the Leverhulme Numeracy Research Programme much more detailed observation of a small case-study sample of the pupils and teachers drawn from the larger sample to try and identify how specific tasks, talk, tools and relationships affect pupil learning.

## REFERENCES

Alexander, P. A. (1991). A cognitive perspective on mathematics: issues of perception, instruction and assessment. In Proceedings of *NATO Advanced Research Workshop: Information Technologies & Mathematical Problem Solving Research* conference, vol. . Oporto, Portugal.

BEST COPY AVAILABLE

Baroody, A. J. & Ginsburg, H. P. (1990). Children's mathematical learning: a cognitive view. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* Reston VA: NCTM.

Brown, J. S.; Collins, A. & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher,* **18**(1), 32-42.

Brown, M.; Denvir, H.; Rhodes, V.; Askew, M.; Wiliam, D. & Ranson, E. (2000). The effect of some classroom factors on grade 3 pupil gains in the Leverhulme Numeracy Research Programme. In Proceedings of *Proceedings of PME 24* conference. Hiroshima: University of Hiroshima.

Carpenter, T. P.; Moser, J. M. & Romberg, T. A. (1982). *Addition and subtraction: a cognitive perspective.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Galton, M. (1989). *Teaching in the primary school.* London: David Fulton.

Kieran, C. (1990). Cognitive processes involved in learning school algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education (ICMI Study Series)* (pp. 96-112). Cambridge: Cambridge University Press.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate Peripheral Participation.* Cambridge: Cambridge University Press.

Peterson, P. L.; Swing, S. R.; Stark, K. D. & Waas, G. A. (1984). Students' cognitions and time on task during mathematics instruction. *American Educational Research Journal,* **21**(3), 487-515.

Rogoff, B. (1990). *Apprenticeship in thinking: cognitive development in social context.* New York: Oxford University Press.

Rogoff, B. (1995). Evaluating development in the process of participation: theory, methods, and practice building on each other. In E. Amsel & A. Renninger (Eds.), *Change and development: Issues of theory, application and method* Hillsdale, NJ: Lawrence Erlbaum.

Saxe, G. B. (1991). *Culture and cognitive development: studies in mathematical understanding.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher,* **27**(2), 4-13.

Wiliam, D. (1992). Special needs and the distribution of attainment in the national curriculum. *British Journal of Educational Psychology,* **62**, 397-403.

# AN ANALYSIS OF BRACKET EXPANSION ERRORS

**Paul Ayres**
**University of New South Wales, Australia**

*This paper analyses the errors made on a set of bracket expansions problems which require the completion of four multiplications. The majority of errors (85%) occurred because signs were incorrectly combined. In particular, 91% of these errors involved at least one minus sign. Other common errors were table-related, involved adding instead of multiplying, or were procedural in nature (bugs). It was also found that errors were more frequent during the second and fourth multiplications, suggesting that working memory load was greatest at these points.*

The discovery of systematic errors in mathematics has long been valued by educators (see Ashlock, 1986; Maurer, 1987; Brown & Burton, 1978). Some researchers (Brown & Burton, 1978; Resnick, Nesher, Leonard, Magone, Omason & Peled, 1989) argue that error analysis is an important diagnostic tool in identifying common misconceptions and wrongly applied strategies. By identifying specific errors, mathematics teachers can then select the appropriate remedial actions to help eliminate these errors. However, the cause of errors can be quite complex, as errors can be caused by limitations in working memory (Kintsch & Greeno, 1985) as well as by lack of knowledge.

## Working memory load and mathematical errors

Considerable research has been conducted into the nature and cause of mathematical errors. Hitch (1978) found that the solution to a mental calculation, such as 347 + 189, was highly dependent upon the problem solver's ability to hold and process information. If any of the initial information or interim subtotals were forgotten, errors would occur. Consequently, Hitch argued that mental arithmetic errors were caused by decay in the storage of problem information. More recent research by Ashcraft, Donley, Halas & Vakali (1992), and Logie, Gilhooly & Wynn (1994), has confirmed the link between calculation errors and loss of information. Furthermore, Campbell and Charness (1990) found that when students were mentally required to square large numbers (such as 74) by using a calculation algorithm, errors had a higher chance of occurring at "heavy traffic stages in the squaring task" (p. 887), arguing that these stages corresponded with the high demands of keeping track of subgoals and their results. In addition, Ashcraft et al. (1992) reported that the retrieval of basic addition facts could also be affected by working memory demands. An increase in problem difficulty corresponded to less accuracy in recalling number facts. This last finding is of some significance as it suggests that working memory load affects the retrieval of information as well as its storage.

Much of the research into the relationship between working memory and errors has been conducted in the domain of mental arithmetic. However, evidence of this relationship has also been found in other areas of mathematics such as arithmetic word problems (Fayol, Abdi & Gombert, 1987) and geometry (Ayres & Sweller, 1990; Ayres, 1993). Fayol et al. employed fairly simple word problems and found that the way that the problem texts were organised led to significant performance differences. Changing the order of the wording of the problems, for example, led to more errors and a reduced performance. Fayol et al. argued that certain text structures forced the participants to problem solve in a manner (bottom up) which overloaded working memory and forced errors. Similarly, studies by Ayres & Sweller (1990), and Ayres (1993) in geometry, found that students made more errors in the calculation of subgoals than on final goal stages. Errors were not necessarily made because the geometry theorems were poorly learnt, but because the subgoal stage demanded more working memory resources and forced errors.

The research reported above indicates that working memory load can be a major cause of errors in a number of mathematical domains. A heavy load not only causes temporary information to be lost but also appears to interfere with the recall and manipulation of information from long term memory (Ashcraft et al., 1992; Ayres & Sweller, 1990). On more complex tasks involving geometry (Ayres & Sweller, 1990) and mental calculation (Campbell & Charness, 1990), distinct error clusters appeared within problems at points where the load was heaviest.

This paper forms part of a study which aims to extend the research on working memory and mathematical errors by investigating errors on elementary algebra tasks. In a previous paper, Ayres (In press) argued that if cognitive load affects problem solving accuracy then error clusters should be found throughout mathematics because problems vary considerably and automatically create uneven cognitive loads irrespective of the problem solving strategies employed. At points in a problem where the most storage and/or processing of information takes place (described as "working memory clutter" by Campbell and Charness, 1990) it is expected that more errors will occur. To test this prediction in an area of algebra the following experiment was completed.

## Instrument Design

In studying algebra, students learn to multiply out (expand) brackets of the type shown below (referred to as *Example-1* ).

$$-3 ( -4 - 5x ) - 2 ( -3x - 4 ).$$

To remove these brackets, the term before each bracket must pre-multiply each term within the bracket. This expansion leads to the calculation of the following four operations, listed in their usual order of completion: $-3 * -4$ (Operation-1) , $-3 * -5x$ (Operation-2), $-2 * -3x$ (Operation-3), and $-2 * -4$ (Operation-4). Typically students work from left to right, calculate one operation at a time and record individual products. Whereas this process might seem straight forward, it is far from trivial for beginning algebra students. Studies have shown that many students have great difficulty understanding the basic concepts of algebra (see Küchemann, 1981; Kieran, 1992; Herscovics & Linchevski 1995; MacGregor & Stacey, 1996) and negative numbers (see Gallardo, 1994; Herscovics & Linchevski, 1995). It is therefore argued that tasks which combine a knowledge of brackets with the manipulation of algebraic expressions and negative numbers are fairly complex and may exert a heavy load on working memory. Furthermore, at points where cognitive load is greatest, errors are more likely to occur as a result of stored information being lost and/or new information being incorrectly retrieved from long term memory.

It should be noted that the expansion of Example-1 always requires the multiplication of two negative numbers together. However, there are four possible combinations involving positives and negatives (+ * +, + * -, - * + and - * -). A problem-set was therefore designed to include the four combinations of signs in order to explore potential differences between them. Furthermore, to investigate whether the occurrence of errors varied over the four operations, the problem set was counterbalanced by using two conditions. Firstly, as negatives were expected to play a significant role in causing errors, the four multiplicative combinations of pluses and minuses were equally distributed over the problem set. In order to achieve this balance, a set of eight problems (see Table 1) was constructed isomorphic to Example-1. To ensure a counterbalanced design, each of the four combinations of signs appeared twice at each operational position, randomly distributed over the set. Secondly, as manipulation of terms containing x could evoke higher cognitive loads than terms without x, the x's were also evenly distributed across the four operations. Each problem contained two x's, one in each bracket. The numbers in the problem set were not counterbalanced in the same fashion as the signs. This was desirable, as otherwise each problem would have looked very similar and the set would have lacked variety. Nevertheless to reduce the possible impact of number fact biases, all calculations involved single digit numbers only. The highest resultant product required was 32, derived from 4 x 8.

Table 1 *The problem set used in the study*

| | | | |
|---|---|---|---|
| Q1. | $5 (3 - 4x) + 3 (-x - 3)$ | Q 2. | $2 (5x + 2) + 4 (-3 + 4x)$ |
| Q3. | $-3 (1 + 3x) - 7 (-2 + x)$ | Q4. | $-4 (4x -8) - 3 (2x + 6)$ |
| Q 5. | $-7 ( -1 - 2x) + 8 (2x + 3)$ | Q6. | $-2 (-2x + 6) - 3 (9 - x)$ |
| Q7. | $2 (-3x - 5) + 3 (4 - 3x)$ | Q8. | $5 (-3 + 5x) - 2 (-4x -7)$ |

## Method

*Participants.* A total of 229 students from Grade 8 (average age of 13.4 years) and Grade 9 (average age of 14.2 years) from three Sydney metropolitan high schools participated. Fifty one percent were girls. All students had some experience in expanding brackets.

*Materials and procedure.* Each student received the set of eight problems (see Table 1) on a single sheet of paper with sufficient space after each problem to complete the required task for that problem. Students were instructed to expand the brackets only, which required the completion of four operations. In the case of Example-1, students were expected to produce the following type of answer:

- 3 (-4 - 5x) - 2 (-3x - 4) = 12 + 15x + 6x + 8.

Having completed the four operations as shown, students were then instructed to start the next problem. Students were not required to simplify further by grouping terms together (to get 21x + 20), as only the first phase in bracket expansions was of interest in this experiment. Calculators were not allowed and sufficient time was given for students to complete the problem set.

## Results

### *Types of errors*

Twenty seven students were excluded from the study because they experienced severe difficulties and did not attempt all eight problems. Their exclusion was necessary because of the counterbalanced nature of the problem-set. A non-solution to a single problem would unbalance the distribution of signs and therefore bias the result. A high proportion of students (38%) made no errors as might be expected in an exercise of this nature involving students with a wide range of abilities. The types of errors made and on which operations they occurred (location) were documented for each student. Errors were initially classified as either sign (84.8% of all errors) or numerical (15.2%). Sign errors occurred if the two signs were multiplied together incorrectly (e.g. -3 * -2 = -6), whereas numerical errors occurred if the numbers were multiplied incorrectly (e.g. -3 * -2 = 9).

In multiplying the two numbers together (excluding signs), specific types of errors were observed. The most frequent (35.2% of numerical errors) category of error was *table-related*, as incorrect answers were found to be multiples of the relevant "times-table" (for example 8*3 = 27). This error has been frequently found during research into arithmetic (see Campbell and Clark, 1992). Twenty six different types of this error were identified. Fourteen of which were incorrect by one multiple (for example 2 * 5 = 15, 8 * 2 = 18, 8 * 3 = 21). The most frequent table-related errors corresponded to 8 * 3. The second most frequent error (25.7%) was forgetting to include the number before the (*pre-multiplier*) during the calculation of a specific operation. For example, a student might expand 5 ( 3 - 4x) as 15 - 4x, where the five has not been included during the second operation. Another major source of error (19%) was *adding* the numbers together instead of multiplying them. For example 8 - 20x might be given as an answer to 5 ( 3 - 4x). Remaining errors included *Interference* (8.6%) where the previous answer would be repeated (e.g. 15 - 15x), *Subtracting* (3.8%) the numbers (e.g. 2 -20x), *missing digits* (2.9%) from the answer (e.g. 5 - 2x), excluding numbers *within* (1 %) the bracket (e.g. 15 - 5x). The remaining errors (3.8 %) were classified as miscellaneous errors as they could not be uniquely classified.

*Analysis of sign errors*

Error clusters were formed over the four operations as the percentage of sign errors made over operations 1, 2, 3 and 4 were 10.5%, 28.8%, 19.2% and 41.5% respectively. It was previously reported by Ayres (In press) that these variations were significantly different. More errors were made during the expansion of the second bracket compared with the first bracket, and more errors were made during the second operation compared with the first operation within each bracket. As the problem set was counterbalanced Ayres argued that these cluster represent differences in working memory load. In particular, most errors were made at the fourth operation which corresponded to the greatest working memory load. Further insights were gained by examining the individual error distributions for each of the sign combinations. The two operations which had pluses as a pre-multiplier (+ * - and + * +) contributed little towards the error count (8.7% of errors). In contrast, operations with a minus as a pre-multiplier ( - * +, - * -) caused the majority of errors (91.3%). In particular, nearly 60% of the errors resulted from the incorrect multiplication of two minuses. Consequently, it was argued that minus signs contribute significantly towards increasing working memory load.

*Analysis of numerical errors.*

The distribution of numerical errors were 13.1%, 37.4%, 19.6% and 29.9% on Operations 1, 2, 3 and 4 respectively, and therefore quite similar to the results found for sign multiplications. As the numbers were not counterbalanced, no statistical tests of significance were conducted. Nevertheless, the observed distribution suggests that numerical errors are generally clustered around the second operation in each bracket. However, over the 32 operations in the problem set, no significant correlation ($r = 0.17$, $p = 0.36$) was found between numerical errors and sign errors. Unlike the sign-error patterns which were greatly influenced by operations including negatives, numerical errors were generally evenly distributed over the combinations of + * + (25.7%), + * - (22.9%), - * + (27.6%) and - * - (23.8%). Whereas, 3 *( 8 + ...) was unlikely to produce a negative answer, an numerical error was more likely to occur.

*Procedural errors (Bugs)*

Brown and Burton (1978) identified a class of systematic errors in arithmetic which they called bugs. Unlike careless errors or slips, bugs were procedural and often featured an incorrect routine in an otherwise correct method. Although little research has been conducted into bracket expansions, Larkin (1989) reported that in expanding -3 (5x + 7), students may exclude the pre-multiplier for the second operation and incorrectly obtain - 15x + 7. For a two bracket example, there are a number of possible bugs. An analysis of individual solutions revealed that eight students had error profiles consistent with excluding the sign before both brackets during the second and fourth operations. In addition, twenty three students made errors consistent with excluding the sign before the second bracket only during the fourth operation. For example, - 3 (-4 - 5x) - 2 (-3x - 4) would be expanded as 12 - 15x + 6x - 8 for the former or 12 + 15x + 6x - 8 for the latter. It was also revealed in the analysis of numerical errors that a common error was to exclude the pre-multiplying number before the bracket during calculation. This error also suggests a procedural problem, although it was not consistently employed by any student.

## Discussion

Distinct error clusters were observed in this study as more errors were made during the expansion of the second bracket compared with the first bracket, and more errors were made during the second operation compared with the first operation within each bracket. It was discovered that many students could easily multiply -3 and -2 together if it appeared during the first operation, but more likely to make an error (-6) if it appeared during the fourth operation. Consequently, it is highly feasible that working memory load was not equally distributed over the four operations. At points where the load was heaviest, information was lost from working memory or incorrectly recalled from long term memory. It has been previously argued by Ayres (In press) that more decision making processes are made on the second operation within each bracket and during the expansion of the second bracket due to the dual-role of signs which link together brackets and operations

This study has also provided considerable information on bracket expansion tasks, some of which may have implications for instruction. Many students were subjected to varying degrees of working memory problems on this task. At the centre of these difficulties was the working memory load associated with the second and fourth operations, as well as the second bracket. Consequently, to reduce cognitive load at these key positions, a greater emphasis may have to be made at these points during instruction. Additional information was also collected about procedural errors. A significant number of students were thought to be making procedural errors by systematically leaving out minus signs. The use of either of the two bugs identified, indicates a particular misconception about bracket expansions which might be directly addressed by making students aware of these errors. Some students also omitted a pre-multiplying number, although this was less frequent. A further common error was table-related. Students would often recall the wrong multiple answer to the product of two numbers. This phenomenon is well known, but is not easily explained or rectified. Campbell and Clark (1992) argue that it is caused by multiple associations with the two numbers. Furthermore, it is clear that many students have difficulties with negative numbers. Consequently, attaining directed number competence should be an important stage before attempting bracket tasks.

Finally, it has been shown here and in other domains, such as geometry (Ayres and Sweller, 1990) and complex algorithms (Campbell & Charness, 1990), that errors can also occur because of increases in cognitive load, rather than because of poor learning. Educators employing error analysis must therefore be careful in their interpretations. Some errors may require additional analysis to distinguish their cause. A potentially useful tool in this role could be the utilisation of a set of counterbalanced problems, such as the set employed in this study. The counterbalanced design was instrumental in identifying the error profiles found in this study. It not only detected working memory problems, but also indicated some likely procedural errors. A similar design was also used by Ayres and Sweller (1990) to detect error discrepancies in geometry. Thus, counterbalanced problem sets might be useful in diagnosing errors in other mathematical domains and have the added advantage of simple classroom applications.

## References

Ashcraft, M.H., Donley, R.D., Halas, M.A., & Vakali, M. (1992). Working memory, automaticity, and problem difficulty. In J.I.D. Campbell (Ed.), *The Nature and Origins of Mathematical Skills (Advances in Psychology "91),* (pp. 301-329), Amsterdam: Elsevier.

Ashlock, R.B. (1986). *Error patterns in computation: A semi-programmed approach, 4th edition.* Columbus, Ohio: Charles E. Merrill Publishing Company.

Ayres, P. (In press). Systematic mathematical errors and cognitive load. *Contemporary Educational psychology.*

Ayres, P. (1993). Why goal-free problems can facilitate learning. *Contemporary Educational Psychology, 18,* 376-381.

Ayres, P. & Sweller, J. (1990). Locus of difficulty in multi-stage mathematics problems. *American Journal of Psychology, Vol. 103,* No. 2, 167-193..

Brown, A.L. & Burton, R.R. (1978). Diagnostic models for procedural bugs in basic mathematical skills. *Cognitive Science, 2,* 155-192.

Campbell, J.I.D. & Charness, N. (1990). Age-related declines in working-memory skills: Evidence from a complex calculation task. *Developmental Psychology, 26,* 879-888.

Campbell, J.I.D. & Clark, J.M. (1992). Cognitive number processing: An encoding-complex perspective. In J.I.D. Campbell (Ed.), *The Nature and Origins of Mathematical Skills (Advances in Psychology "91),* (pp. 457-491), Amsterdam: Elsevier.

Fayol, M., Abdi, H. & Gombert, J.E. (1987). Arithmetic problems formulation and working memory load. *Cognition and Instruction, 4,* 187-202.

Gallardo, A. (1994). Negative numbers in algebra: The use of a teaching model. In J.P. Da Pointe & J. F. Matos (Eds.). *Proceedings of the XIX Annual*

*Conference for the Psychology of Mathematics Education.*, Lisbon, Portugal, Vol. 2, 376-383.

Herscovics, N., & Linchevski, L. (1995). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27, 1, 59-78.

Hitch, G. J. (1978). The role of short-term working memory in mental arithmetic. *Cognitive Psychology, 10*, 302-323.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, NY, NY: Macmillan Publishing Company.

Kintsch, W. & Greeno, J.G. (1985). Understanding and solving word arithmetic problems. *Psychological Review, 92*, 109-129.

Küchemann,, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16*. Murray: London, 102-119.

Larkin, J. H. (1989). Robust performance in algebra: The role of the problem representation. In S. Wagner & C. Kieran (Eds.). *Research issues in the learning and teaching of algebra, Vol. 4,* (pp. 120-134). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum Associates.

Logie, R.H., Gilhooly, K.J & Wynn, V. (1994). Counting on working memory in arithmetic problem solving. *Memory and Cognition*, 22, 395-410.

MacGregor, M. & Stacey, K. (1996). Progress in learning algebra: Temporary and persistent difficulties *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education.*, Valencia, Spain, Vol. 3, 297-304.

Maurer, S.B. (1987). New knowledge about errors and new views about learners: What they mean to educators and more educators would like to know. In A.H. Schoenfeld (Ed.). *Cognitive Science and Mathematics Education*. Hillsdale, NJ: Erlbaum.

Resnick, L.B., Nesher, P., Leonard, F., Magone, M., Omanson, S. & Peled, I. (1989). Conceptual bases of arithmetic errors: *The case of decimal fractions. Journal for Research in Mathematics Education*, Vol. 20, No. 1, 8-27.

# KNOWING THE SAMPLE SPACE OR NOT: THE EFFECTS ON DECISION MAKING

**Paul Ayres** **Jenni Way**
**University of New South Wales** **University of Cambridge**
**Australia** **UK**

*Grade eight students observed a video-recording of coloured balls being drawn from a box with replacement. After every fifth selection, students were required to predict the colour of the next ball drawn. It was discovered that knowing or not knowing the sample space made no difference to predictions. However, students were significantly influenced in their probability judgements by confirmation or refutation of their own 'predictions'. Furthermore, interviews revealed that many students inappropriately tried to utilise colour patterns as a strategy.*

Much research into probabilistic reasoning has featured the effect of sequences of randomly generated outcomes on the subject's expectations regarding the 'next' outcome. One such strategy that has received considerable attention is *representativeness* (Fischbein & Schnarch, 1997; Shaughnessy, 1981), which is the expectation that a random set of outcomes should be representative of the composition of the known sample space. In particular, Shaughnessy (1981) found that college students believe that some combinations of sequences are more likely than others, even though statistically they have the same chance of occurring. For example, in flipping a coin, HHTHTH is considered more likely than HHHHTT. Amir, Linchevski & Shefet (1999) found a similar effect with 11 and 12 year-olds. Furthermore, Amir et al. argued that

> "The 'representativeness' heuristic includes two distinct and independent dimensions: the tendency to expect a sample to reflect the numerical proportion of the parent population; the tendency to expect a sample not to be too orderly, to look 'random'. (p.2-32).

Closely related to representativeness is the type of thinking known as *negative recency* (Fischbein & Schnarch, 1997) where there exists the expectation that as the frequency of a particular outcome increases the probability of that outcome occurring decreases. For example; when repeatedly flipping a coin, a run of heads would lead to the expectation of the next flip being a tail.

A lot of this research has been conducted with adults, using written tasks or 'tests', in which preconceived sets of outcomes have been presented to the subjects. However, researchers working with children often prefer to use real random generators to accommodate children's need for concrete experiences (for example: Truran, 1992; Way, 1996). A method often used is to produce a number of experimental outcomes, then ask the children to state what they consider to be the most likely outcome of the next random event. This context gives rise to a little studied, possible influence on decision-making; that of the confirmation or refutation of the 'prediction' by the actual next outcome.

In a study by Truran (1996), working with a *known sample space*, the changes in prediction of primary and secondary students in regards to the next outcome were analysed. One finding was that when the more-likely outcome was predicted, it didn't really matter whether the next outcome confirmed or refuted that prediction. However, if the less-likely outcome was predicted, the subject was highly likely to change the prediction, particularly if the following outcome refuted the less-likely prediction. Similarly, Ayres & Way (1998b, 1999) working with *unknown sample spaces*, found evidence that upper primary-aged students would change their prediction patterns according to how successful they were in their predictions. Although, students would choose the most frequently occurring colour under specific conditions, they would change strategy if their predictions were not rewarded. Consequently, Ayres & Way (1999) argued that children might be influenced in their probability judgements by confirmation or refutation of their 'predictions' rather than the overall picture.

This study was designed to extend the work of Ayres and Way on the influence of confirmation or refutation of predictions by including older students (grade 8). Also, the effect of knowledge about the sample space on predictions was also measured, as their previous experiments did not feature known sample spaces. Furthermore, students were interviewed in order to gain additional insights into their decision-making processes.

## METHOD

*Participants.* Sixty grade-eight students (13 year-olds) from a large secondary high school in the State of New South Wales, Australia, participated in this study. No students had been formally taught probability theory in their mathematics classes. These students were in the top two mathematics classes of the grade and therefore many would rank above the State average in general mathematical ability.

*Apparatus.* To test the effects that the influence of prediction confirmation has on students, two video recordings were made of coloured balls being selected from a box. Both featured a presenter making thirty selections from the box with replacement. As previously described by Ayres & Way (1998a,1999) the outcomes were manipulated to form two specific sequences. In spite of pre-ordained outcomes, the videos were made in such a way as to appear authentic. Consequently, it was expected that students viewing the videos would believe that the draws were random (see Ayres & Way, 1998a). In both sequences, 19 whites (63%), 7 blues (23%) and 4 yellows (13%) made up the set. However, there was a significant difference between the two sequences. For one sequence, the most frequently occurring colour (white), appeared consistently (four out of five times) when a prediction was made. This sequence was called the *Typical Outcome* sequence (see Table 1). Students who predicted a number of whites would therefore be relatively successful. For the second sequence, the less likely outcomes (blue and yellow) appeared consistently (four out of five) at the prediction locations. This sequence was called the *Non-typical Outcome* sequence. To achieve this second sequence, the colours in the first sequence in the prediction positions (6th, 11th, 16th, 21st, 26th) were rotated with a different colour within the same subset of five colours (see Table 1). As a result, only one white appeared in the first five prediction positions. Students who predicted a number of whites would therefore not be very successful. However, the accumulated experimental probabilities after each set of five outcomes for both sequences were identical, and approximately matched the theoretical probabilities of the intended sample space of 6 white, 3 blue and 1 yellow ball. Consequently, if students viewed either sequence and were guided by experimental probabilities alone, they would choose white as the most likely outcome in both situations. However, depending upon which sequence was viewed students' success rated would vary considerably. It was therefore anticipated that this design would measure how students' predictions would be influenced by the success rate of their own previous predictions.

*Table 1: The two colour sequences*

| Outcomes in sequences of five | Typical Outcome Sequence | Non-typical Outcome Sequence | Experimental Probabilities (%) W : B : Y |
|---|---|---|---|
| First | WYWBW | WYWBW | 60 : 20 : 20 |
| Second | WWBWY | BWWWY | 60 : 20 : 20 |
| Third | BWWWB | WBWWB | 60 : 27 : 13 |
| Fourth | WYBWW | YWBWW | 60 : 25 : 15 |
| Fifth | WBWWW | BWWWW | 64 : 24 : 12 |
| Sixth | WWYBW | YWWBW | 63 : 23 : 13 |

Note: Predictions made *after* each fifth draw(underlined).

*Procedure.* Students were randomly assigned to four groups. Two groups were assigned a *typical outcome* sequence and two groups were assigned the *non-typical outcome* sequence. To test the effects that knowing or not knowing the sample space has on students, a further sub-division was completed. One group from each of the assigned *typical* and *non-typical* sequences groupings were told exactly what the sample space was (i.e. 10 balls, of which 6 were white, 3 blue and 1 yellow). The remaining two groups were only informed that there were some white, blue and yellow balls in the box, but no clue was given to the proportions. The experiment was conducted with small classes of students each time. All students were told that it was a game and that they should try to predict as many correct colours as possible. When the experimenter was satisfied that the students understood the nature of the task, they were shown one of the videos according to which group they had been assigned. After each selection a record of the colours was also recorded on the classroom chalkboard. After the first five colours were observed being drawn, the video was stopped and students were asked to make their predictions on their answer sheets. This procedure was then repeated for five more subsets. The task was identical for each group.

*Interviews.* Shortly following the completion of this task, three students were chosen at random from each group and interviewed. During this process, each student were shown their responses to the previous task as the sequence unfolded in sub-sequences of five colours. After each prediction they were asked to explain their choice of colour. Their answers were recorded by the interviewer.

## RESULTS

### Quantitative Data

For each student, a sequence consisting of six colour predictions, was recorded. Given the nature of the prediction tasks in this study, of particular interest was the number of whites chosen. If students were guided by experimental probability then it was expected that a high percentage of whites would be chosen. To investigate this, the mean number of whites chosen for each group was calculated (see Table 2). In addition, the number of whites chosen in the first and last three predictions was also recorded (see Table 2). Ayres and Way (1998b, 1999) found that students may not necessarily choose the most frequent colour after a small number of observations and may need more information before committing to a strategy. By comparing predictions over the two halves it was possible to analyse the extent to which students refined their strategies as more selections were observed.

*Table 2: Mean number of Whites selected by each group*

| | Typical Outcomes | | | Non-typical Outcomes | | |
|---|---|---|---|---|---|---|
| Predictions | Ratio Known | Ratio not Known | Comb. Totals | Ratio Known | Ratio not Known | Comb. Totals |
| | (n = 14) | (n = 17) | | (n = 15) | (n = 14) | |
| First 3 | 1.8 | 1.8 | 1.8 | 2.2 | 1.8 | 2.0 |
| Predictions | (0.9)* | (1.0) | (0.9) | (0.9) | (0.9) | (0.9) |
| Second 3 | 2.2 | 2.2 | 2.2 | 1.7 | 1.1 | 1.4 |
| Predictions | (0..8) | (0.9) | (0.9) | (1.0) | (0.9) | (1.0) |
| Overall | 4.0 | 4.1 | 4.0 | 3.9 | 2.9 | 3.4 |
| Predictions | (1.4) | (1.4) | (1.4) | (1.3) | (1.5) | (1.5) |

*Note: Standard deviations are given in brackets.

Two-way ANOVAs were completed on the data. The two main effects consisted of knowledge of the *sample space* and the nature of the observed colour *sequence*. Over the total number of predictions, there was no sample space main effect, F (1, 56) = 1.71, p = 0.20, or interaction, F (1,56) = 2.16, p = 0.15. However, there was a significant sequence effect at the 93% level, F(1, 56) = 3.37, p =0.07. For the first three predictions, there was no significant main effects or interaction (all p values were greater than 0.34). For the last three predictions, there was a significant sequence effect, F(1, 56) = 13.38, p = 0.001; but no sample space effect, F(1,56) = 1.51, p = 0.23, or interaction, F(1, 56) = 1.74, p = 0.19.

It is clear from the above analysis that knowledge of the sample space did not have an effect in the experiment. In contrast, the colour sequence observed had a dramatic effect on student predictions. For the last three predictions, the mean (combined) number of whites chosen for the *typical* sequence groups was 2.2 which was significantly higher than combined mean, 1.4, for the *non-typical* sequence groups. As previously found by Ayres & Way (1999) the groups which were being rewarded with correct answers by choosing the most likely outcome, increased the use of this strategy; whereas, the groups which were not being rewarded moved away from the strategy. Knowledge of the sample space did not help the group which received an *non-typical* colour sequence. Although over the last three predictions, this group's mean (1.7) was higher than the *non-typical* group's mean (1.1) which had no knowledge of the problem space. However, this difference was not statistically significant.

*Prediction Patterns.* To gain further insights into the type of strategies employed, prediction profiles for each group were found by calculating the frequency of each colour selected at each prediction point (data not reported due to space restrictions). It is interesting to note that three groups chose a high percentage of yellows for their first selection This may be an example of the *negative recency* effect as the other two colours had more recently occurred (see Table 1). The major between-group differences emerged at the fourth and fifth prediction stages as the two groups who received an untypical sequence chose white less than 50% of the time. At the sixth prediction, there was also some notable differences. Both typical-sequence groups mainly chose white (86% and 88%). Similarly, the *non-typical* group who knew the sample space, predicted a high percentage (80%) of whites. In contrast, the *non-typical* group who did not know the sample space chose more blues (43%) than whites (36%).

### Qualitative Data

Summaries of each interview are given in Table 3. Eight of the students (2, 4, 5, 6, 9, 10, 11 & 12) appeared aware that white was the most frequently occurring colour and chose white a number of times. Interestingly, four of these students used the word *safe* when choosing white. This may indicate a good understanding of likelihood by being cautious. One student (10) selected white for the first five selections before switching to blue. This student remarked that she "couldn't understand why more white weren't coming up". Obviously, this student who received the *non-typical* sequence, expected white to occur more often, and through frustration switched to a different colour. Seven students (five from the *non-typical* group) were influenced by patterns which they detected in the sequences. In many cases, a search for patterns was continued throughout the whole trial. There was also a number of occurrences (students 1, 3, 4, 5, 6, 7, 8, 9, 12) of the negative recency effect which often involved the selection of yellow. Finally of note, only two of the six students who knew the sample space chose white in their first prediction.

*Table 3: Summary of student interviews.*

| | |
|---|---|
| ***Know sample space- Typical outcomes*** | |
| S1 | 1.-(B)* Only a few Y, more chance of B. 2.-(B) same as before, one Y so far more chance of a B. 3-(W) Between Y and W, more chance of W coming out. 4-(Y) Gave Y a go. 5.-(W) Between B and W, more W therefore higher chance. 6.-(B) 3W in row so chose colour I chose all along. |
| S2 | 1.-(W) Chose W as I thought balls were left out of box. 2.(W) The first 3 were W so I thought next one would be W. 3.-(B) I'm not sure as B being picked a bit more. 4.-(W) More W are coming so I chose W. 5.-(W) More W. 6-(W) Playing safe. W will probably come out |
| S3 | 1.-(Y) Pattern WY, WB. 2.-(B) Lots of W, but thought another colour should come out. 3.-(Y) A Y hadn't been selected for a while. 4-(W). Don't know. getting sick of wrong answer. 5-(B) For every W I chose, I then chose another colour. 6-(W). My turn to chose W. |
| ***Don't Know sample space - Typical outcomes*** | |
| S4 | 1.-(Y) Seemed to be more Y. 2.-(B) Already picked so many W, chose Y already so chose B. 3.-(Y) No Y so far so picked it. 4.-(W) Realised more W so I selected W. 5.-(W) More W. 6.-(W) Playing safe. W will probably come. |
| S5 | 1.-(W) Chose W as more of them. 2.-(Y) Some kind of pattern 2W 2Y, so picked Y. 3.-(Y) Hadn't been a Y in this sequence. 4.-(W) Seems to be more W. 5.-(W) More W. 6.-(W) More W. |
| S6 | 1.-(Y) Pattern of WB WY. 2.-(W) White often came out after a colour. 3.-(W) More white in the box. 4.-(Y) Blue was last colour to be selected. 5.-(W) More whites are being selected. 6.-(W) More W. |
| ***Know sample space- Untypical outcomes*** | |
| S7 | 1.-(B) No idea. B favourite colour. 2.-(B) Didn't realise B was correct- Don't know why I chose B again. 3.-(Y) Hadn't seen Y so far. 4.-(W). I saw a pattern. B seemed to be coming out after 2W 5.-(Y) Looking for a colour after 2W. 6.-(W) There seems to be 2W in a pattern, so I chose W. |
| S8 | 1.-(B). Favourite colour 2.-(W) not much Y and I knew there were six W. 3.-(Y) Y hadn't come up yet. 4.-(W) I don't know why I picked W. 5.-(Y)I thought Y would come along again.. 6.-(W). W seems to come up twice. |
| S9 | 1.-(W) Seemed to be more W. 2.-(W) Best to stick with W. 3.-(W) Between Y and W, more chance of W. 4.-(Y) Gave Y a go. 5.-(W) More W. 6.-(W). Stick to W. |

* Note: Record of colours chosen.

*Table 3 (continued).*

| ***Don't Know sample space- Untypical outcomes*** | |
|---|---|
| S10 | 1.-(Y) Pattern. Already picked B so chose Y. 2.-(W) More whites in the box. I saw no pattern, so safe to chose W. 3.-(W) More W are obvious. process of elimination. Deduced B as well so I chose W. 4.-(W) More W and so W is safer. 5.-(W) Heaps more W, so I chose it.. 6.-(B) Thought a colour should be next. Couldn't understand why more W weren't coming up. |
| S11 | 1.-(Y) I saw a pattern. 2.-(W) Began to see more W. 3.-(W) Gone for the pattern again. 4.-(W) Gone for the majority. 5.-(B) B seems to be more common than Y. 6.-(W) Majority still W. |
| S12 | 1.-(Y) On the trial there were more W, so I picked Y. 2.-(W) More W coming out. 3.-(W) Pattern WB, go with W to keep pattern. 4.-(B) Blue next choice. 1 B in 5. 5.-(W) More W balls, it is safer to choose W. 6.-(W) More W- it is safer to choose W. |

## CONCLUSIONS

For the particular colour sequences utilised in this study, knowing or not knowing the sample space did not produce significant differences, indicating that many students failed to use this information for a basis for their predictions. In contrast, the order of colours in the two sequences made a major impact. Groups who observed the *typical* sequence chose white more consistently than groups who observed the *non-typical* sequence, especially in the second half of the trials. This latter result is consistent with the earlier findings by Ayres & Way (199b, 1999) with younger students and suggests that a wide age-range of students may be influenced in their probability judgements by confirmation or refutation of their predictions.

Of further interest was the discovery that many students used strategies based on patterns. There are two feasible explanations for this. Firstly, as Shaughnessy (1981) and Amir et al. (1999) reported, students may believe that some combinations of sequences are more likely than others, even though statistically they have the same chance of occurring. Secondly, the students in this study have experienced mathematics curricula (in common with many other states and countries) which have placed a considerable focus on the importance of patterns. It may be that be that their experiences with patterns have encouraged them to try to identify them in this study. If so, it creates a possible conflict between the understanding of chance and an integral aspect of mathematics. At this stage it is not clear whether these two explanations, either separately or in combination, explain the use of a pattern strategy. Therefore, the use of patterns in this domain needs to investigated further in future.

Finally, it should be noted that these students would have considerable experience with some of the basic concepts (ratio and proportion) which are crucial to understanding elementary probability. This experience may have helped some students who showed quite sophisticated understanding of chance. Nevertheless, many exhibited the same misconceptions as the upper primary-aged students reported by Ayres & Way (1998b, 1999), such as negative recency. Consequently, it appears that two or more years of mathematics education has not strengthened the understanding of likelihood for many. Once again, the results of the study have shown the need for an earlier introduction to official instruction into probability, which is not introduced in the New South Wales curriculum until grade 9.

## REFERENCES

Amir, G., Linchevski,L. & Shefet, M. (1999). The probabilistic thinking of 11-12 year old children. In Zaslavsky, O., (Ed.) *Proceedings of the Twenty-Third Conference of the International Group for the Psychology of Mathematics Education.* Vol. 2, (pp. 25-32), Haifa, Israel.

Ayres, P & Way, J (1999). Decision-making strategies in probability experiments: the influence of prediction confirmation. In Zaslavsky, O., (Ed.) *Proceedings of the Twenty-Third Conference of the International Group for the Psychology of Mathematics Education.* Vol. 2.(pp. 41-48) Haifa, Israel.

Ayres, P & Way, J (1998a). The use of a video-recorder in probability experiments. In Pereira-Mendoza, L., Kea, L.S., Tang, W.K., & Wong, W-K. (Eds). *Published Proceedings of the Fifth International Conference on Teaching Statistics (ICOTS-5),* Vol 3, p 1415 - 1421. Singapore, June 1998.

Ayres, P & Way, J (1998b). Factors influencing predictions made on randomly generated sequences. In Kanes, C., Goos, M. & Warren, E. (Eds.). *Published Proceedings of the Mathematical Education Research Group of Australasia (MERGA 21),* p 74 - 81. Gold Coast, Australia, July 1998.

Fischbein, E. & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal of Research in Mathematics Education, 28*: 1, 96-105.

Shaughnessy, J. M. (1981). Misconceptions of probability: From systematic errors to systematic errors and decisions. In A. Shulte & J. Smart, *1981 Yearbook: Teaching Statistics and Probability.* (pp.90-100.) Reston, Vi.: NCTM.

Truran, J. (1992). Children's understanding of the independence of random generators. In B. Southwell, B. Perry & K. Owens (Eds.) *Space - The First and Final Frontier: Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia.* (pp.541-550). Richmond, Australia.

Truran, J. (1996). Children's misconceptions about the independence of random generators. In L, Puig & A, Gutierrez (Eds.), *Proceedings of the Twentieth International Conference for the Psychology of Mathematics Education.* Vol. 4.(pp. 331-338) Valencia, Spain.

Way, J. (1996). Children's Strategies for Comparing Two Types of Random Generators. In L. Puig and A. Gutierrez (Eds.) *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education.* Valencia, Spain. (pp. 419-426).

# The Development of Mathematics Education Based on Ethnomathematics (2)
## - Analysis of Universal Activities in terms of Verbs -

**Takuya BABA**
**Graduate School for International Development and Cooperation**
**Hiroshima University**

**Hideki IWASAKI**
**Graduate School for International Development and Cooperation**
**Hiroshima University**

### Abstract

*In the development of new curriculum whose focus is mathematical activity as signifié, the verb as its signifian should be a center of consideration. This is why the proposed curriculum is named verb-based. In mathematics education, an activity deepens itself in a recursive manner through symbolization of activity. Bishop (1991) widened this concept of activity and claimed each culture has developed its own mathematics through six universal activities.*

*All the verbs at the primary level in the Japanese course of study were collected and analyzed in this paper. As a result this research showed the structure of activities on mathematical recognition through the verbs under the theory of internalization.*

## 1. Six universal activities and verb-based curriculum

In the teaching and learning process of mathematics classroom, an interaction develops among three components, children, teacher and mathematics. Among these components, children play an important role in the constructivism that is based on the Piaget's epistemology and has developed its theory by focusing on their activity. According to its theory, the operation is interpreted as an activity to be internalized as a result of action towards environment, and furthermore mathematics as a system of such operations.

When we regard children's activity as an action to act upon their own environment, the environment should include not only mathematical objects but also other objects, which are highly related to their own culture. Action upon the objects does not occur in the vacuum of abstractness. Even mathematical objects of today have been created through many years by abstracting some properties of objects found in the environment. In this sense, focusing on activities in this paper means to seek for the possibility of widening up the range of objects.

From different perspective, Bishop (1991) argues that each culture has developed mathematics and there exist *six universal activities* to be observed at the basement of these mathematics. For example, there may be different counting systems and expressions of numbers from culture to culture, but there is no culture that does not have a counting activity. In this sense six activities possess a kind of universality but at the same time their various appearances indicate the existence of sub-activities depending on their characteristics. It is there that possibility lies for the analysis of the inter-structure of various cultural activities.

Course of study usually has an arrangement and structure of knowledge as a target to achieve. More weight used to be given to the knowledge acquired at the end of class and so naturally a main concern of mathematics education was the formula and technique to enable the speed and accuracy of solution. However recently this tendency has been under criticism, and instead the attitudinal aspects of the process such as interests, willingness and attitudes, have been put more stress upon. If the curriculum is based upon the knowledge, which can be expressed in noun form, then it may be called noun-based curriculum. On the other hand, the curriculum to be considered in this paper is called verb-based curriculum whose concern is activities in the sense of the constructivism.

Naturally, it is important to have actual activities in mind in accordance with verbs listed in this curriculum. In the teaching-learning process there are such activities as interaction between teacher and children, mutual interaction among children, and an individual activity by children. And their details may differ from one another depending upon time, place and after all culture. The authors have assumed, however, that the essence of activity is expressed in terms of verbs. This paper is a first step towards the development of this new curriculum principle.

From the above discussion, the objectives of this paper are as follows;

(1) To clarify characteristics of verbs,

(2) To analyze the verbs in the course of study and their inter-structure,

(3) To develop theoretical foundation for verb-based curriculum.

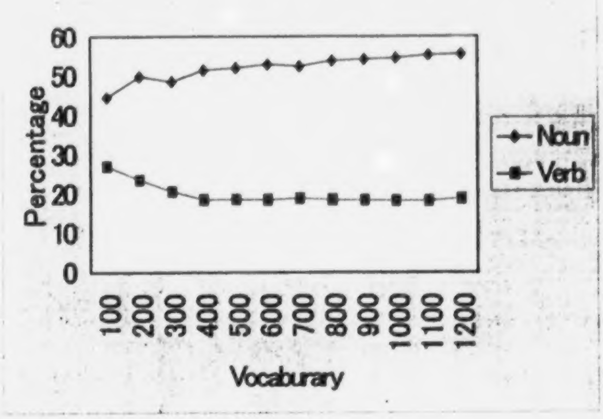
## 2. Characteristics of verbs

The general characteristics of verbs are analyzed before the analysis of the verbs in the course of study. Like shown in the below table and graph, nouns occupy more percentage of the vocabulary than verbs and that percentage goes even higher as the number of vocabulary, which are most frequently used, increases in Japanese language. This, in other words, indicates that one verb can take many nouns as an object, a place, means and so on. Of course a noun can take a certain number of verbs as well, but the verb has a large versatility considering the number of objects. For example the verb 'to count', which is typical in mathematics education, can take such objects as cows, papers and many others. The counting action for cows and the one for papers differ in an appearance, but the word 'to count' is abstracted as a common property from those activities.

The reason why verbs as signifiant are fewer than nouns, lies in the property of activity as signifié that is instantaneous and does not retain its locus of movement very long. For example, the activity 'to hit' can be perceived by means of eyes and ears, but it only remains as an afterimage for a while and in the next moment its existence cannot be perceived by our senses no longer.

The another property of verb is flexibility towards object to take. The way of hitting a door and the way of hitting a shoulder are different in terms of an object of activity, movement of hand and even the part of hand to use. And the intention of activity also differs from each other. While the former is to call an attention of somebody inside the room, the latter is to alleviate stiffness in the shoulder or to inform somebody about demotion to the lower post in a certain context. Since these two activities are differentiated not in terms of verbs but in terms of nouns, even when a new object emerges, the activity towards it can be expressed in analogy with the existing activities. For example, the computer-related facilities are fast permeating into our daily life and the emergence of new objects such as keyboard, mouse and so on urges an extensive interpretation of the existing activities and a widened usage of the corresponding verbs in order to cover a new activity.

| Vocabulary | Noun | Verb |
|---|---|---|
| 100 | 44.5 | 27 |
| 200 | 49.8 | 23.5 |
| 300 | 48.5 | 20.5 |
| 400 | 51.4 | 18.4 |
| 500 | 52 | 18.5 |
| 600 | 52.9 | 184 |
| 700 | 52.4 | 18.8 |
| 800 | 53.8 | 18.4 |
| 900 | 54.2 | 18.3 |
| 1000 | 54.5 | 18.2 |
| 1100 | 55.3 | 18.2 |
| 1200 | 55.5 | 18.7 |



**Pic.1 the graph of the percentage of noun and verb**
**Table1 vocabulary and percentage of noun and verb(left):National Language Institute**

Three characteristics regarding verbs have been discussed so far. They are in short 'the comparatively small number of verbs', 'the activity as signifié of verbs is instantaneous' and ' the flexibility of verbs towards its objects'. This last one can integrate all the characteristics of verbs and represent the manifoldness in verb's association with other words, especially nouns. Then this characteristic is to be named an 'elasticity' of verb to accommodate newly created objects in analogy with elasticity of rubber that absorbs a shock coming from outside.

### 3. Analysis of Verbs in Japanese Course of Study

The present course of study in Japan has been considered, though the analysis is limited only to the teaching content of the primary school. The below table shows only verbs for grade 1 as an example. A, B and C in the below table show domains in the course of study. They are namely Numbers and Calculations, Quantities and Measurements and Geometric Figures.

| | Verb | | Quasi-verb |
|---|---|---|---|
| A | to represent(表す) | to understand, to know(理解する) | understand(理解) |
| | to compare(比べる) | to interpret, to read(よむ) | correspondence(対応) |
| | to count(数える) | to group(まとめる) | |
| | to know(知る) | to make(作る) | manipulation(操作) |
| | regarding(みる) | to be able to, to be applied(できる) | carry out(計算) |

BEST COPY AVAILABLE

| | | |
|---|---|---|
| | N.A.(する)<br>to use, to be applied(用いる)<br>to consider … in relation(関係付ける)<br>to divide… into equal parts(等分する)<br>in a neatly arranged way(整理する) | meaning(意味) |
| B | Through(通す)<br>to compare(比べる)<br>to read(よむ)<br>around(ある)<br>N.A.(できる)<br>N.A.(する)<br>N.A.(なる) | comparing(比較)<br>experiences(経験)<br>measurement(測定)<br>manipulation(操作)<br>understand(理解) |
| C | Through(通す)<br>to decompose(分解する)<br>to recognize(認める)<br>to use(用いる)<br>to grasp(とらえる)<br>to represent(言い表す)<br>to construct(作る)<br>N.A.(する)<br>N.A.(なる) | observing(観察)<br>experiences(経験)<br>constructing(構成)<br>manipulative activities(操作)<br>understand(理解) |

**Table2 Verbs for grade 1 in the course of study**

(NOTE1: Domain D, Quantitative Relations, starts at grade3.)

(NOTE2: Originally the verbs are collected and analyzed in Japanese, which are written in the dictionary form and placed inside the brackets, and then translated into English with reference to Japan Society of Mathematics Education (1990), which are placed in front. Some of them are no longer verbs nor correspond with one set of words. (e.g. through(通す), to understand, to know(理解する)) Because of this, both Japanese and English are provided for reference.)

(NOTE3: Some Japanese verbs do not have English translation but since their meanings are insignificant, they are excluded from further analysis.)

The quasi-verb in this table is categorized as a noun in Japanese that becomes a verb when it is suffixed by 'suru', which means 'to do'. It strong association with verb can be easily known from English translation. It is a combination of two Chinese characters that are originally either both verbs or a pair of verb and noun, but as a whole it has a verb-like meaning.

There are several points noted from this analysis of verbs.

(1) There are many transitive verbs.

There are in total 68 verbs that appear in the course of study, if each verb is counted only once. Only 8 verbs out of them are intransitive verbs. While intransitive verbs express certain *state* of things, transitive verbs represent '*activity towards objects*' which is fundamental in mathematics education. And the large number of transitive verbs manifests its focus on activities.

(2) Making activity an object of another activity

* Usage of masu form of verb as a noun

(Example) duplications(重なり), dispersion(散らばり) etc.

* How-to

(Example) how to represent(表し方) etc.

For this same Japanese, there are different English translations, that are the way of representing, notation, how ..are set, representation.

* Quasi-verbs

(Example) manipulation, measurement etc.

They are related to *objectification of activity*. This means the activity itself is converted to an object of another activity by changing somehow the verb to a noun.

(3) The role of verb 'to deepen'

There are 6 verbs which are common to all four domains. Namely they are 'to use', 'to interpret', 'to understand', 'to represent', 'to know' and 'to deepen'. The last verbs has special function in such a sense that the represented activity develops based on the knowledge that has been learned before. Its first appearance for domains A, B, C and D in the course of study is 2nd grade, 3rd grade, 3rd grade and 5th grade respectively. They can be regarded as boundary between basic and advanced topics.

(4) There are 20 verbs that belong to both domains A and D.

This number constitutes approximately 60% of verbs in domain D. It is known from this that activities in domain D develop on the basis of activities in domain A.

As a summary these four points describe that much attention is given to activities towards objects and to how to develop them. Now it is an imperative task to structure activities and thus verbs to attain our goals.

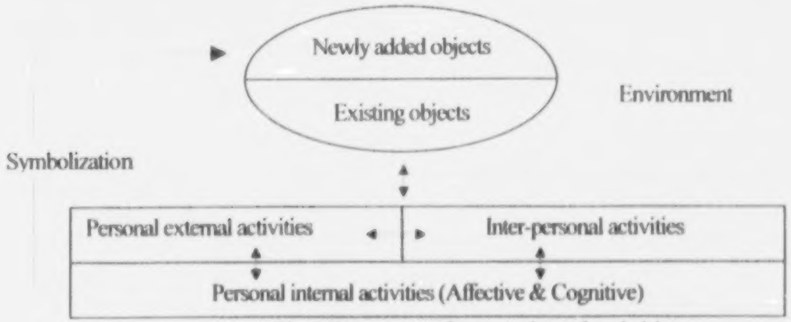
### 4. From the structure of activities to the one of verbs

The 68 verbs extracted in the previous section are analyzed further in this section. They consist of 60 transitive verbs and 8 intransitive ones. We excluded from the former category the verbs that do not take human being as a subject and the ones that have only specific objects. It is because the purpose of this research is to consider children's activities towards objects. Finally the 51 verbs are selected and analyzed.

Considering the nature of activities, they are categorized into personal ones and inter-personal ones, and the former is further subdivided into the ones which involve external action and the ones which do not. The categorization of verbs virtually follows that of activities. Naturally all groups of activities are related to each other. However, if more than two categories are involved in certain activity, then the preference was given to the category of personal and external activities, because this category among all represents children's activities towards objects most explicitly and thus is the most crucial for the verb-based curriculum.

Now the six universal activities are revisited for further analysis. The two 'to explain' and 'to play' belong to the category of inter-personal activities and the remaining four, 'to count', 'to measure', 'to locate' and 'to design' belong to the category of personal and external activities. Of course the activity to count may be exercised among several persons in a certain context, but it is regarded as a personal activity because of the same reason as in the above.

Bishop (1991) analyzed that the activity 'to explain' belongs to the meta-level of the other activities. This interpersonal activity, to explain is related closely with the personal internal activity 'to understand'. The latter activity develops within oneself but the former gives a boost to its development. In other words the activities to understand and to explain are dual process which interact and stimulate each other, and the clog in this interaction process causes a reflection towards the activity.


Newly added objects
Existing objects
Environment
Symbolization
Personal external activities
Inter-personal activities
Personal internal activities (Affective & Cognitive)

Pic.2 Structure of categories of activities

To play seems to have a lot of interesting implications for teaching-learning process but their clarification is a future task in mathematics education. It has two aspects of personal and social activities but in any case it does not put much importance on the outcome of that activity. Enjoyment is its main proponent and in this sense it is uniquely related to the affective side of learning activity.

While all the six activities may possess aspect of creation, the activity to design has a rigorous part in this.

> "Designing involves imagining nature without the 'unnecessary' parts, and perhaps even emphasizing some aspects more than others. To great extent, then, designing concerns abstracting a shape from the natural environment." (Bishop, 1991, p.39)

This is very important not only from mathematical point of view, but also cultural point of view. While most of our activities are based on the environment that already exists, the creative aspect of activities adds a new entry into the list of objects in the environment.

From this analysis of universal activities, the above structure is drawn. And this structure is projected onto that of verbs. Through this whole process, mere action is changed to the activity with intention, the activity upon objects is then internalized into operation, and this internalized operation, by being given a symbol, becomes a new entry into the environment.

### 5. Towards theoretical foundation of verb-based curriculum

By interpreting an activity extensively in the verb-based curriculum, the objects can be easily extended to cultural ones beyond those abstracted in the modern Western society. It is from this same perspective that ethnomathematics interprets mathematics as a cultural endeavor and has posed arguments about mathematics especially in the mathematics education (D'Ambrosio;1985, Gerdes;1990). It also provides substance to the verb-based curriculum in terms of objects, activities, and symbolized activities existing within cultures. Mathematics as a culture, however, should not mean to remain in the past or the present, but beyond this line the verb-based curriculum has potentiality to absorb new entries with 'elasticity of verbs'.

The previous research (1998) discussed it is important both to view mathematics and mathematics education critically with ethnomathematics and to view ethnomathematics itself critically by application of three elements, that are critical competence, critical distance, and critical engagement.

As mentioned above, ethnomathematics shares the same base as the verb-based curriculum that the scope of mathematics should be widened in terms of activities and objects. Thus the first half of the claim is already mentioned here to view mathematics and mathematics education critically. On the other hand, although to learn techniques of cultural activities and to practice them is a starting point and basis for next steps, it is definitely not enough to remain there. So objectification of activity is an essential part of the curriculum. And the latter half, the critical review of ethnomathematics, is incarnated in this part of internalization process.

In summary it is important to base oneself within a culture, to create a new object through objectification and symbolization of cultural activities and at the same time to view the object critically. This is the same recursive process as structure of activities discussed in the previous section. The critical thinking is a propeller of such process.

In Gerdes (1990, 1999), sona was presented and analyzed for mathematics education. Of course in school mathematics, just to be able to draw many patterns of sona cannot be an objective. There are many indicative hints, however, in this example not only for children in the Angola where they were originally drawn but also for 'demathematized members in the highly mathematized society' (Keitel, 1998). The children within Angolan community of course feel mathematics closer to them and thus cultural tie. Even the children in the developed country may be able to appreciate other culture and so it enables them to view other possibilities of design, to grow interest in other patterns and to create new shapes. It all requires widened view of mathematics.

Picked from the domain C in the Japanese course of study, the activities 'to construct', 'to draw' and 'to decompose' are practiced at the beginning as primary activities. Such activities as 'to observe' and 'to represent' may give stimulus to some ideas and the children may be able 'to consider', 'to recognize' and 'to pay attention to' some important aspects and to objectify primary activities. The ideas may not be very new or innovative, but what the children find in their activities is essential in the teaching/ learning process. And as a result of these activities, they may internalize and symbolize primary activities and have secondary activities such as 'to investigate' and 'to deepen' acting upon them. This process is recursive in such a way that any previous activities can be considered by the succeeding activities through their objectification.

As future directions of this research we have plans;

(1) To analyze the verbs in the Kenyan syllabus and thus activities and to make comparative study between the cases of Japan and Kenya,

(2) To design and to analyze classroom practice using verb-based curriculum.

In this paper mathematical activity is assumed to be internalized into the individual learner's schema, but this assumption itself should be examined as a part of future research. Thus, how the primary activities are internalized and symbolized and how the symbolized activities are considered as a part of secondary activities constitute a main part of the analysis for classroom practice.

After all, in the above example of sona drawing, the children may have a sense of accomplishment by practically creating new drawings. This creativity is in depth linked with human's potentiality to reflect activities critically and grasp them systematically. In the verb-based curriculum the focus on cultural activities is directed towards the development of this fundamental nature of human beings, mathematics.

**References**

Baba,T., Iwasaki,H.(1998) 'The Development of Mathematics Education Based on Ethnomathematics: The Intersection of Critical Mathematics Education and Ethnomathematics', *Proceedings of 23rd PME*, Haifa, Israel.

Bishop,A.J.: 1991, *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*, Kluwer Academic Publishers.

D'Ambrosio,U.(1985) 'Socio-cultural Bases for Mathematical Education', *Proceedings of 5th ICME*, Adelaide, Australia.

Habermas,J.:1971, *Knowledge and Human Interests*, Beacon Press.

Gerdes,P.(1990) 'On Mathematical Elments in the Tchokwe "Sona" Taradition', *For the Learning of Mathematics*, Vol. 10, pp.31-34.

Gerdes,P.: 1999, *Geometry from Africa: Mathematical and Educational Explanations*, The Mathematical Association of America.

Iwasaki,H., Nakumra,S., Baba,T.(1999) 'The Present Significance of Media Literacy and its Development: Focusing on the Science Education development', *The Bulletin of Japanese Curriculum Research and Development*, Vol. 22, No. 1, pp.43-53.

Japan Society of Mathematics Edcuation (1990) *Mathematics Program in Japan: Kindergarten to Upper Secondary School.*

Keitel,C.(1997); 'Perspective of Mathematics Education for 21st Century – Mathematics Curricula: For Whose Benefits?', *Plenary Speech at JSME 30.*

Ministry of Education:1989, *Course of Study, Primary Education*, Ministry of Finance Press (in Japanese).

Nakahara,T.:1995, *Constructivism Approach in Mathematics Education*, Seibunsha (in Japanese).

National Language Institute:1984, *Research and Education of Vocabulary*, Ministry of Finance Press (in Japanese).

Skovsmose,O:1994, *Towards a Philosophy of critical Mathematics Education*, Kluwer Academic Publishers.

# Maths as social and explanations for 'underachievement' in numeracy.

**D A Baker with B V Street**
**School of Education,**
**University of Brighton, UK**

## Abstract

*This is a discussion of research in the 'Schooled and Community numeracies' focus within the Leverhulme funded Low Educational Achievement in Numeracy Research Programme. The intentions of the research in this focus are to seek explanations for underachievement in numeracy that derive from understandings of maths as social. We want to understand why some children engage easily with home numeracy practices yet struggle with schooled numeracies. We want to investigate boundaries children face or which are constructed between home and schooled numeracy practices. The paper will consider some of the conceptual and methodological issues that have arisen in the research.*

## 1 Introduction

This paper is a discussion of one part of the Leverhulme Numeracy Research Programme (1997 – 2002) investigating Low educational achievement in numeracy in the UK. The intentions of the programme are to contribute to understandings of: progression in primary maths; how classroom practices affect attainment; training and intervention strategies; teacher change; and effects of social factors. This paper will concentrate on the latter aspects, which is the prime concern of focus 4 of the Programme on *'Schooled and Community Numeracies'*. The team of researchers working on this focus are Dave Baker from the University of Brighton, and Brian Street and Alison Tomlin, from Kings College, London. The intentions of the research in this focus are to seek explanations for underachievement in numeracy that derive from understandings of maths as social. The reasoning behind this approach is that research on, and educational policy for, raising achievement in numeracy has recently mainly focused on 'school effects' such as teacher subject knowledge, pedagogy (pace, style, whole class teaching, setting, calculators, homework), schools (leadership, effective management, policies), and educational structures (assessment regimes and systems, Local Education Authorities). Consideration of 'home effects' and other social factors have been marginalised and rejected despite serious research exposing their significance. For instance:

> "The data from the National Child Development Survey (1991) show that there is a strong relationship between children's performance in maths and reading tests between the ages of six and eight and their parents' earnings, with the children of higher earning parents performing better." (Machin, 1999 p 19).

Macro visible social factors like poverty clearly do play a large role in educational achievement. In terms of numeracy we want to focus on less conspicuous social factors. We want to understand why some children apparently cope easily with formal numeracy practices whilst others struggle to handle those practices. We want to investigate boundaries and barriers children face between home and school numeracy practices; how these boundaries are constructed and maintained. The paper will initially unpack some of the conceptual and methodological issues that have arisen in the research so far. Field work from 1999 will be used to throw further light on these issues. Implications for research and for teaching will conclude the paper.

## 2 Conceptual issues

The objective of looking at maths as 'social' is to understand and describe different meanings that pupils bring to their encounters with schooled numeracy and thereby to contribute to understandings and explanations for the underachievement of many in schooled maths. This includes ways of viewing maths practices in social contexts that might complement and enrich current tools and language for description available in the field. The concepts we needed to clarify were: understandings of numeracy as social; the nature of home and schooled numeracies; and home/community/school relationships. A great deal of work in the field of 'social literacies' has addressed many of these issues. One dimension of the research is to consider how far this 'social literacies work' (cf. Street, 1996 and Heath, 1982) can be applied to the field of maths education.

The first aspect we needed to clarify was the social in numeracy. A social perspective on maths does not entail simply privileging everyday or 'ethnic' maths or treating everything as 'social', which can be rather vacuous. Nor does it entail pronouncing on the ontological status of maths. Rather it provides a vehicle for an exploratory inquiry into what follows from considering maths as social practice. In my recent research into this question, (Baker, 1999), I found that the dominant understanding of the 'social' in maths was very narrow. It was limited to either interactions with others or to a functional or useful role for numeracy. As an alternative I proposed a different, broader view, which sees the social in terms of ideology and discourse, power relations, values, beliefs, social relations and social institutions, (Baker, 1999). Here, 'values and beliefs' feature in choices made and in contexts in which numeracy is sited. The contexts of home and school are very different and in this research we need to understand the extent to which the numeracy practices sited within them were different. 'Social relations' refer to positions, roles and identities of individuals in relations to others in terms of numeracy. For example we are positioned as insiders or outsiders in a community of numeracy practitioners. 'Social institutions' and procedures are to do with issues of control, legitimacy, status and privileging of some practices over others in maths, as evidenced through accepted and dominant paradigms and procedures. The narrow view which is based on an autonomous model of numeracy as described by Baker and Street, (1996) together with conventional pedagogy and curriculum (DfEE, 1999c), leads to blaming failure or underachievement in numeracy on the teacher, the child or the home and seeing them in some sense in deficit: the teacher in terms of her subject knowledge or her use of ineffective teaching practices; the child in her lack of skills, knowledge and understandings; and the home as lacking the schooled numeracy knowledge to support their children (Freebody and Ludwig, 1996). The broader social model, which makes the epistemological and ideological explicit (Baker and Street 1996), provides different ways of viewing and understanding underachievement and could lead to policies that go beyond access and empowerment towards transformations of curriculum and pedagogy. Instead of viewing underachievement in terms of deficit in dominant practices the model accepts social notions of difference and multiple practices and seek to represent and build upon informal numeracy practices and funds of knowledge, (Moll, 1992 and Heath, 1982).

One concept that flows through the above discussion and through our work is that of numeracy practices. As has been discussed elsewhere, (Baker, 1996), and in parallel to literacy practices, (cf. Street, 1996), we see numeracy practices as more than behaviours that occur when people do maths. We propose that numeracy practices include the conceptualisations, the discourse, the values and beliefs and the social relations that surround these activities as well as the context in which they are sited. The concept of numeracy practices is grounded in the broad notion of the social in maths and is a central concept in our research. It provides a language of description and a lens through which to view practices in different contexts, and leads to an acceptance of multiple numeracies, each one framed and sited in the context in which it occurs.

We are particularly interested in relationships between home/community numeracy and schooled numeracy practices. These relationships are about ways these practices are the same or different, the boundaries between them and the ways they are viewed. Work on the former can be seen in the work of Massingila et al (1996), Baker (1996) or Abreu (1995) and in literacy by Heath (1982). It is argued that these numeracy practices are different because of the context, the values or the discourses in which they are sited. It is suggested in these articles that no set of practices is superior but that they are different. On the other hand, it is also clear that each is constructed and viewed quite differently. Schools and educational policy privilege schooled numeracy over home practices, seeing relationships between them as unequal with the role of homes subservient to that of schools and the boundaries between them clearly delineated. Homes are places where the numeracy practices of the school are to be practised and reinforced. Homework is set by the schools and it is hoped that 'the home' will assist the children's schooled numeracy activities. In a document on the National Numeracy Strategy the DfEE states:

> "An important part of the NNS is that parents are involved and well informed about their children's learning at school. Before parents can help their children effectively with maths, they need to understand something of how maths is taught in school". (DfEE 1999a).

Tasks set for the children to do at home are based on the needs of the school. The DfEE (1999b, p 20) provide sample tasks to be tackled in homes. These assume parental involvement which may not be appropriate for all homes or may even be rejected by some parents. For example, a parent at Mountford School on the project, (cf. fieldwork below), when questioned about homework and home based tasks said:

> "the Government has got it wrong. Children have other things to do at home. Maybe they should stay at school for 20 min and finish it off. Then go home and play. My dad used to ask me if I had any. I would say no so I could go out and play". (6 July 99)

An alternative view which sees the home as possible sites of rich educational resources or as 'funds of knowledge' (Moll, 1992), are not seriously considered. Yet this might in the long term have much to offer as a possible strategy to raise achievement in maths.

### 3 Methodological Issues

In seeking to investigate differences between schooled and home numeracies and other relationships between home and school we need to study events in classrooms and in schools in some depth. In the light of this we are basing our work on detailed case studies. This throws up two key methodological issues: firstly, selection of schools and individual children; and secondly, accessing home numeracy practices. The development of a language of description for identifying and categorising numeracy practices in both schooled and community contexts is an issue as well as deciding what precise practices and data the research would seek and collect. We need to develop case studies of home practices and decide on the balance of interviewing and observing. Some of these issues are discussed below others are raised in the account of the fieldwork.

The sites for the case studies have been chosen to be contrastive, 'telling' cases (Mitchell 1984). The criteria for selection enable us to cover, where possible, the main dimensions of suspected heterogeneity in the population. We selected three schools according to social features frequently cited as significant for achievement in schools. These were location (Freebody et al 1996), ethnicity (Jones, 1998), and relative affluence (Machin, 1999). One of the schools is in a mainly 'white', affluent suburb, the second has a 'white' socially deprived catchment area, and the third is in a mixed urban area attended by predominantly 'black' children. Four children will be chosen initially from one class in each of the three schools, a total of 12 children. The children will be recruited from reception classes, as nearest in social influence to the home environment, and followed through reception to years 1 and 2. The children will be selected in consultation with the teacher. To suit our contrastive methodology we will seek children with the greatest differences in home/school relationships. This will be difficult in practice and we intend to use more observable features such as parental involvement in the school/community, relative family affluence or pupil attainment as indicators of these differences and to give us a contrast in the successful or relatively unsuccessful negotiation of boundaries and barriers between home and schooled numeracies. A substantial constraint on selection will be the agreement or non-agreement of 'parents' to be part of the research.

Data will include: field notes from observations of school lessons, home numeracy activities and 'community' numeracy practices and events amongst pupils from these schools; collections of work and texts used in those contexts, including official curriculum documents, course documents, 'homework', teacher feedback materials; documents regarding home/school links; audio-recorded interviews with teachers, parents and pupils. We are also drawing upon documents on home/school relationships from the NNS. The balance of school and home visits will need to be flexible as it will build reflexively on previous visits and will depend on access to homes.

## 4 Fieldwork.

The purpose of the research is to investigate relationships between home and schooled numeracy practices. To do this we need to investigate numeracy practices in both home and schools together with the views of teachers, parents, schools, children and official documents on the interrelationship. So far we have investigated some numeracy practices in schools and have begun to visit homes as well. What I describe below is a discussion of differences in practices that arose between home and school on the use of games together with interpretations that have emerged so far.

In 1998/9, within a pilot phase of the project we worked for 6 months with children aged 4 to 5 years old in Mountford School, a primary school serving a 'white' socially deprived housing estate with high unemployment. We observed and participated in classroom activities, mainly associated with numeracy but unavoidably also associated with literacy. We then visited selected children at home to contrast their schooled practices with their homes practices.

In one instance of this kind I worked with a group on a literacy activity using an game on a number track, like 'Snakes & Ladders'. The track was in the form of a winding snake marked out in stepping stones with a letter in each section. The game, similar to snakes and ladders involved the children taking turns rolling a die, moving their pieces along the track, and then stating the name of an object beginning with the letter in the section. The winner was the first player to get to the end of the track. The children hardly engaged with the literacy aspects of the game at all. They found the activity hard not because the literacy demands were difficult for them but because they struggled with gaming and numeracy concepts contained in the activity. Game playing required acceptance and understandings of turn taking, rolling a die, the purpose of the rolling of a die, ways of relating the sign on the die to amount of movement along the track, which direction to move along the track, where to move from, which parts of the track counted as a unit, and a desire to win. They had to accept that they had to stay on a section till their next turn. They frequently wanted to handle and play with their pieces between turns. The relationships between different representations of numbers - dots, numerals and moves along the track were not obvious to them either. This suggested that the numeracy practices that the teacher wanted to draw on for this task included those associated with game playing and with number tracks. One could represent this as a lack of skills and the children as in deficit. An alternative perspective was to question whether game playing was in fact part of these children's home experiences and whether the children's unfamiliarity at home with cultural activities like game playing may help explain the children's difficulties or even 'under-achievement'.

We decided to investigate the use of games on number tracks at home and visited one of the children. It was clear that he had never played games like snakes and ladders that used many of these gaming and number track concepts. This is not to say that he did not play other games. In fact he spent over an hour playing a fantasy game with a 40 cm tall model of Godzilla, a large two legged dinosaur. On the ground next to its feet were 3 cm high cars and people made to scale. This model had a wide range of numeracy practices hidden in it including, comparisons of heights, power, size and scale, language as well as fantasy and creativity. When I offered to play snakes and ladders with him at home he showed no interest. His numeracy practices at home were different from those at school. His practices were not necessarily less valid, less interesting or less powerful. They were simply different. However, the assumptions of schooling are that children are exposed to game and number track numeracy practices at home and that homes where children do not meet such activities are in deficit. In this case it meant that a child from this particular home was unable to engage fully in the literacy task set for him because for him home and school practices were clearly at odds with each other. That is, boundaries between these practices are more substantial for some children than for others.

Experiences in the pilot study established some methodological practices that we would need to follow during the project. Access to homes will be problematic but could be achieved using both the teacher and the children as a means of introduction. However, longer periods of access over three years as is intended in the project may depend more on being seen to provide homes with aspects of formal educational value in return, (Civil, 1999). Access to schools and teachers are not problematic, perhaps because home/school links are currently important issues.

Issues about home/school relationships may revolve more around teachers' expectations and images of homes. Teachers' views of homes vary. At times they are very positive seeing homes as contributing to the development of "these three delightful lively and beautiful children" (Comment by a teacher after visiting the home of an prospective nursery pupil, 6 Jan 2000). Yet at others they can be more dismissive seeing the same homes as in deficit. The homes are too noisy, unclean and do not provide appropriate support for the children's learning. They do not see the width of possibilities for building on 'funds of knowledge' from home communities. These differing views create conflicting expectations of homes for teachers. On the one hand the homes, particularly from low socio-economic backgrounds, are seen as unable to support the kinds of numeracy practices they would welcome in their classes. And on the other hand, they assume that children's cultural experiences do equip them to use games like snakes and ladders in the classroom. In a parallel way, parents' expectations of schooling are that children will learn formal numeracy practices at school and that children's informal or non-schooled numeracy practices will not be of value in the classroom. The differences between these views of what counts as numeracy and the practices associated with it may go some way to explaining difficulties some children have in moving from home to school. The fieldwork so far has confirmed and extended our concepts of numeracy practices as contextually defined and supported our models of many different numeracy practices. It also confirmed our view that relationships between homes and schooling are asymmetric with schooled numeracy having a high status and home practices being marginalised.

## 5 Implications

When researching social factors and classroom practices there are strong expectations and desires for answers to practical classroom issues. This can result in the neglect of research and theoretical issues that may be of value and interest. At this stage in our research we do not yet think we have answers, we are still investigating the situation. However, where there are some initial implications we see them as relating to teaching; research and policy.

For researchers there are different ways of seeing achievement and underachievement in numeracy. From one perspective there is a deficit model where children need to be enabled by teachers to learn dominant numeracy skills. A different social perspective provides understanding of social aspects of numeracies and multiple numeracy practices, the importance of context, social relations and ideology, and social factors in schooled and home numeracies. Instead of the dominance of deficit and hierarchical models of numeracy practices this perspective proposes notions of difference, multiple practices and of funds of knowledge.

The pedagogical and curricular implications for teaching from a broad view of the social in maths remain a complex question. Our reading of research literature suggests that a move to acceptance of maths as social does not necessarily result in changes in ways of teaching maths. There is not a 1-1 causal relationship between epistemological models and pedagogy. We are, however, suggesting, that broader views of the social in maths can lead to greater understandings of classroom interactions and though such understandings to changes in classroom practices. What our data do tell us is that relationships between home and school are complex and the extension of schooled numeracy into homes through homework or parent evenings, though encouraged in official policy statements, may be problematic. Instead a commitment to making use of the funds of knowledge in homes, acceptance of the value of home numeracy practices, investing resources and energy into identifying and understanding such funds of knowledge, in and out of school, may have more to offer curricula, teachers and schools than has previously been accepted.

Official government concerns about the access to powerful knowledge such as schooled numeracy, particularly for children from educationally disadvantaged homes, may have to be challenged and replaced by transformations in curriculum and pedagogy rather than only in homes. One implication of viewing maths in general and numeracy in particular in this way may be that *both* home and schooling have to change if we are to have any substantive and long lasting affect on achievements in schooled numeracy.

(An earlier version of this paper was submitted to MES2 Portugal, for March 2000)

**References**

Abreu Guida De (1995) "Understanding how Children Experience the Relationship between Home and School Maths" in *Mind Culture and Activity* 2(2).

Baker D A (1996) "Children's Formal and Informal School Numeracy Practice" in D Baker, J Clay & C Fox (eds.) *Challenging Ways of Knowing in English Maths and Science.* London: Falmer

Baker D A (1999) *What it means to understand mathematics as social* ms presented to BCME 4 the British Congress of Maths Education, Northampton.

Baker D & Street B (1996) "Literacy and Numeracy Models" in *The International Encyclopaedia of Adult Education & Training,* Oxford, Elsevier Science, p 79-85

Civil M. (1999) *Parents as Learners of Maths.* ms. civil@math.arizona.edu

DfEE (1998) *The Implementation of the National Numeracy Strategy. Final Report of the Numeracy Task Force.* Sudbury, Suffolk: DfEE

DfEE (1999a) *National Numeracy Strategy at home with numbers.* Watford: BEAM

DfEE (1999b) *The NNS. Sample Homework Activities.* London: DfEE

DfEE (1999c) *The National Numeracy Strategy. Framework for teaching mathematics from Reception to Year 6,* . London: DfEE

Heath S B (1982) "Questioning at Home and at School: A Comparative Study" in G Spindler (ed.) *Doing the Ethnography of Schooling* London: Holt Rinehart Winston

Jones L (1998) "Home and School Numeracy Experiences for Young Somali Pupils in Britain", in *European Early Childhood Educational Research Journal* 6(1)

Machin S (1999) "Childhood Disadvantage and Intergenerational Transmission of Economic Status" in *CASE Report 5 Persistent Poverty and Lifetime inequality: The Evidence,* London: London School of Economics.

Masingila J. O. Davidenko, S. and Prus-Wisniowska E, (1996) "Mathematics Learning and Practice In and Out of School: a framework for connecting these experiences", in *Educational Studies in Mathematics*, Vol. 31 p 175 – 200

Mitchell C (1984) "Typicality and the Case Study" in Ethnographic Research; a guide to general conduct" in R Ellen (ed.) *ASA Research Methods in Social Anthropology,* Academic Press.

Moll L. Amanti C, Neff D and Gonzalez N, (1992) "Funds of Knowledge for Teaching: using a qualitative approach to connect homes and classrooms", in *Theory into practice,* XXXI (2) p 131-141

Street B V (1996) *Social Literacies.* Cambridge: Longman.

# YEAR 6 STUDENTS' IDIOSYNCRATIC NOTIONS OF UNITISING, REUNITISING, AND REGROUPING DECIMAL NUMBER PLACES

Annette R Baturo & Tom J. Cooper
Centre for Mathematics and Science Education, QUT, Brisbane, Australia

*Having flexible notions of the unit (e.g., 26 ones can be thought of as 2.6 tens, 1 ten 15 ones, 260 tenths, etc) should be a major focus of elementary mathematics education. However, often these powerful notions are relegated to computations where the major emphasis is on "getting the right answer" thus procedural knowledge rather than conceptual knowledge becomes the primary focus. This paper reports on 22 high-performing students' reunitising processes ascertained from individual interviews on tasks requiring unitising, reunitising and regrouping; errors were categorised to depict particular thinking strategies. The results show that, even for high-performing students, regrouping is a cognitively complex task. This paper analyses this complexity and draws inferences for teaching.*

Decimal numbers are among the most complex and important mathematical ideas that children will encounter in their mathematics education and students' difficulties in acquiring an understanding of decimal numbers have been well-documented in the literature (Baturo, 1999; Baturo & Cooper, 1997; 1999; Behr, Harel, Post, & Lesh, 1992; Bigelow, Davis, & Hunting, 1989; Resnick et al., 1989; Ross, 1990; Stacey & Steinle, 1999; Wearne & Hiebert, 1988). Behr, Lesh, Post, and Silver (1983) claimed that decimal numbers form "a rich arena within which children can develop and expand the mental structures necessary for intellectual development" (p. 91).

However, having the appropriate knowledge available may not mean that this knowledge is accessed when required (e.g., Prawat, 1989), and correct performance alone is an insufficient indicator of understanding (e.g., Leinhardt, 1988).

This paper reports on students' responses in an interview to reunitising tasks similar to those given in a pencil-and-paper test to determine: (a) the thinking strategies that are employed in tasks such as these; and (b) whether students who have apparently constructed the appropriate knowledge can access this knowledge.

***Mathematics background***. In her study of students' acquisition of, and access to, the cognitions required to function competently with decimal numbers, Baturo (1999) tested 173 Year 6 students from two schools with a pencil-and-paper diagnostic instrument to determine the students' available knowledge of the numeration processes (number identification, place value, counting, regrouping, comparing, ordering, approximating and estimating) for tenths and hundredths. As a result of analyses of the students' performances and of the cognitive components embedded in decimal-number numeration processes, Baturo developed a numeration model (see Figure 1) to show these cognitions and how they may be connected.

The model depicts decimal-number numeration as having three levels of knowledge that are hierarchical in nature and therefore represent a sequence of cognitive complexity. *Level 1 knowledge* is the baseline knowledge associated with *position, base* and *order*, and without which students cannot function with understanding in numeration tasks. Baseline knowledge is unary in nature comprising static memory-objects (Derry, 1996) from which all decimal-number numeration knowledge is derived. *Level 2 knowledge* is the "linking" knowledge associated with *unitisation* (Behr, Harel, Post & Lesh, 1992; Lamon, 1996) and *equivalence,* both of which are derived from the notion of base. It is binary in nature and therefore represents relational mappings (Halford, 1993). *Level 3 knowledge* is the structural knowledge that provides the superstructure for integrating all levels and is associated with *reunitisation, additive structure* and *multiplicative structure.* This level involves ternary relations that are the basis of system mappings (Halford, 1993) and which are much more complex than binary relations. Of interest to this paper is the process of *reunitising* and its foundations, particularly *unitising and equivalence.*

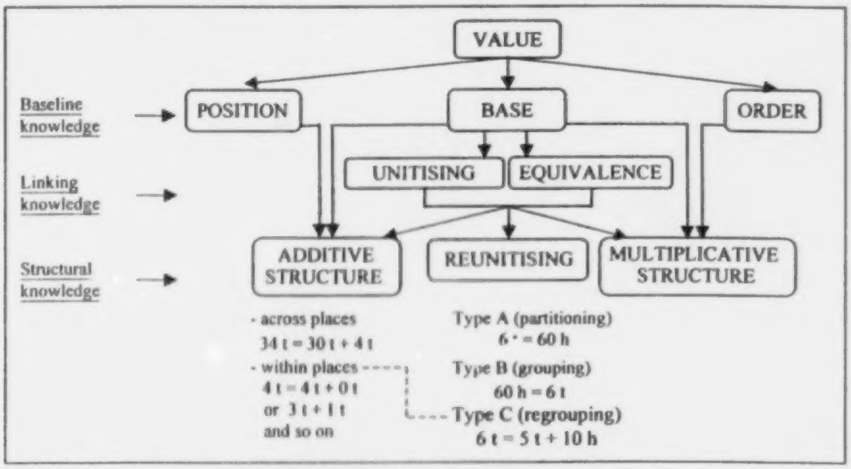

VALUE
Baseline knowledge
POSITION
BASE
ORDER
Linking knowledge
UNITISING
EQUIVALENCE
Structural knowledge
ADDITIVE STRUCTURE
REUNITISING
MULTIPLICATIVE STRUCTURE
- across places
34 t = 30 t + 4 t
- within places
4 t = 4 t + 0 t
or 3 t + 1 t
and so on
Type A (partitioning)
6 t = 60 h
Type B (grouping)
60 h = 6 t
Type C (regrouping)
6 t = 5 t + 10 h

*Figure 1.* Cognitive components and their connections in the decimal number system (Baturo, 1999).

***Reunitisation.*** Baturo & Cooper (1997) identified three types of reunitisation (see Figure 1) which they believe represent a sequence of cognitive complexity, namely: (a) Partitioning to make smaller units (e.g., 6t = 60h); (b) Grouping to make larger units (e.g., 60h = 6t); and (c) regrouping involving additivity (e.g., 6t = 5t + 10h).

Along with *unitisation* and *equivalence*, reunitisation underlies the decimal-number numeration processes of *naming* (number identification/unitisation), *renaming* (partitioning and grouping reunitisation) and *regrouping.*

Reunitisation processes have an application in pencil-and-paper computation, in mental computation, and in calculator computation involving large numbers. However, in paper-and-pencil computation, teaching for proceduralisation of these notions seems to take precedence over the conceptualisation of the processes. On the other hand, flexible notions of the unit are required to develop the number sense

(Heirdsfield, 1999; Sowder, 1988) essential for mental and large-number calculator computation. For example, a calculator will work on, and display, just eight digits; so, in the examples such as those shown in Figure 2, students need to be familiar with the additive structure of the subtraction operation to know that the subtraction can be separated into two subtraction operations (see Figure 2 A). However, regrouping knowledge is also required to do Type B in Figure 2.

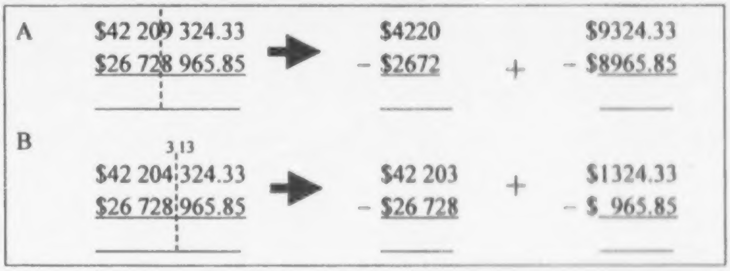


*Figure 2.* Number sense (including regrouping) required for large-number calculator computation

## Method

The 22 Year 6 students reported on in this paper were the highest-performing students selected from 173 Year 6 students from two large Brisbane schools. Selection was made as a result of a diagnostic test instrument that incorporated 109 items related to the numeration categories of number identification (unitisation), renaming (reunitising), counting, comparing, ordering, approximating and estimating. The items, which involved decimal numbers limited to tenths and hundredths, were presented in pictorial, language and symbolic forms. The first 9 of these 22 students were those who scored ≥90% on the diagnostic test; the next 7 students scored from 85-89%; the remaining 6 students scored from 80-84%.

Two instruments were used – the diagnostic test, and a semistructured individual interview schedule. The test was used to determine students' decimal-number numeration strengths and weakness whilst the interview was designed to probe these strengths and weaknesses. The complete interview schedule included tasks related to place value, multiplicative structure and reunitising. However, this paper reports on only one of the reunitising tasks (see Figure 3).

*Contingent questions.* It was expected that some students would be confused by the 2nd interview question (e.g., whether to write 0 tenths or 60 tenths) and, if so, the item would be reworded as: *I know there are no tenths in the tenths place, but how many tenths are in the entire number*? If students couldn't proceed with (c), it would be reworded as: *If I put 59 there* (tenths), *what would you need to write here* (hundredths) *to keep the same value*?

During the first couple of interviews, the students exhibited some doubt as to whether 60 tenths 7 hundredths had the same value as 60.7 (the number they had just reunitised). Therefore, each student was asked, after reunitising, whether the reunitised number had the same value as the original number – *If you put the 60 tenths 7 hundredths together, would you get this number (6.07)*?

| Diagnostic test | Interview schedule |
|---|---|
| Write the missing numbers.<br>(a) 2.09 = ___tenths ___ hundredths [Reunitising]<br>(b) 0.52 = 4 tenths _____ hundredths [Regrouping] | 6.07 = ___ tenths ___ hundredths<br>___ tenths ___ hundredths<br>(a) Read the number. [Unitising]<br>(b) Write the missing numbers [Reunitising – 1st row]<br>(c) Can you write another pair of numbers that have the same value? [Regrouping] |

*Figure 3.* Test and interview reunitising and regrouping tasks relevant to this paper.

Specifically, the interview tasks were to determine whether the students could unitise (identify the fraction as hundredths), reunitise the ones as tenths, and then regroup the tenths and hundredths (processes that can occur in pencil-and-paper subtraction computation e.g., 6.07 – 2.38).

Students were withdrawn from the classroom and interviewed individually; each interview took approximately 50 minutes and wasvideo-taped. The interview results were scored in terms of correctness and thinking strategies were categorised.

## Results

Table 1 provides the interview and test results with respect to*unitising, reunitising*, and *regrouping*. The interview results for regrouping are presented in two parts– *open* and *closed*. The open regrouping task results were the initial responses given by the students without prompting; the closed regrouping task results were the responses given when students were given 59 tenths and they had to supply the number of hundredths. The students are presented on the table in the order of their overall test performance so that the highest performing students (≥90%) are at the top of the table and the lowest performing students (80-84%) at the bottom of the table. However, they weren't interviewed in this order. Actual responses are provided in parentheses.

***Unitising and reunitising***. Table 1 shows that the students had no problems with unitising the number and could generally reunitise the number appropriately (as they had in the test item). The contingent question guiding students to consider the entire number (not just the tenths place) seemed to help all but three students (Roger, Dennis, Sally). When challenged as to the correctness of the answer, Sally changed her answer to 60 tenths 7 hundredths whilst Dennis changed his to 67 tenths. Dennis appears to have developed the rule "zero means nothing" so that 607 has the same value as 67. Roger's protocol is provided. (I = Interviewer)

I: *Does the 7 part belong to the tenths or the hundredths*? [No response] *Write the number that has 67 tenths*. [Roger wrote 6.7.] *Is this the same as this* (pointing to 60.7)? [No] *Show me the tenths place* (in 6.07). [He pointed to the zero.] *So if we go across to the tenths place* (underlining 6.0)*like you did here* (indicating an earlier task, 6 = __ tenths, which he had done correctly) …[R: Oh, I had the 7.] *Yes, but the 7 doesn't belong to the tenths, does it? Could you fix it now*? [Roger inserted a decimal point in his original response, 67 tenths, to make 6.7 tenths.] *Read what you've got.* [Six and seven tenths.]

Table 1

*Students' Responses for the Interview and Test Items in Terms of Correctness.*

| | INTERVIEW | | | | TEST | |
|---|---|---|---|---|---|---|
| | Unitising | Reunitising | Regrouping | | Reunitising | Regrouping |
| | | | Open | Closed [59] tenths | | |
| Paul | ✓ | ✓ | ✓ (50,107) | ✓ | ✓ | ✓ |
| Linda | ✓ | ✗ (607 t) | ✗ NR | ✗ (59,48,8) | ✓ | ✓ |
| Rufus | ✓ | ✓ | ✗ (50,17) | NA | ✓ | ✓ |
| Roger | ✓ | ✗ (67t) | ✗ ND | ✗ ND | ✓ | ✓ |
| Cindy | ✓ | ✓ | NA | ✓ | ✗ (20,209) | ✓ |
| Aldyth | ✓ | ✓ | ✗ (600,-) | ✓ | ✓ | ✓ |
| Sean | ✓ | ✓ | ✓ (0,607) | ✓ | ✓ | ✓ |
| Joe | ✓ | ✓ | ✗ NR | ✓ | ✗ (20,209) | ✓ |
| Noel | ✓ | ✓ | ✗ NR | ✗ (59,107) | ✓ | ✓ |
| Chloe | ✓ | ✓ | ✗ NR | ✗ (59,107) | ✓ | ✓ |
| Jim | ✓ | ✓ | ✗ (50,7) | ✓ | ✓ | ✓ |
| Leon | ✓ | ✓ | ✗ (6.0,7) | ✓ | ✓ | ✓ |
| Meg | ✓ | ✓ | ✓ (0,607) | ✗ (59,8) | ✗ (20,209) | ✗ (4,-) |
| Simon | ✓ | ✓ | ✗ (6,-) | ✗ (59,8) | ✓ | ✓ |
| Hayley | ✓ | ✓ | ✗ (060,7) | ✗ (59,70) | ✓ | ✗ (4,20) |
| Mary | ✓ | ✓ | ✗ NR | ✗ (59,1) | ✓ | ✗ (4,3) |
| Sally | ✓ | ✗ (607,0) | ✗ (0,7) | ✓ | ✗ (0,9) | ✗ (4,2) |
| Julian | ✓ | ✓ | ✓ (0,607) | ✓ | ✓ | ✓ |
| Donald | ✓ | ✓ | ✗ (6,7) | ✓ | ✗(19,10) | ✓ |
| Dennis | ✓ | ✗ (607; 67) | ✗ ND | ✗ ND | ✗ (29,209) | ✗ (4,52) |
| Patsy | ✓ | ✓ | ✗ (87,6) | ✓ | ✓ | ✓ |
| Laura | ✓ | ✓ | ✗ NR | ✗ (59,8) | ✓ | ✗ (4,2) |
| | **100%** | **81.8** | **4/21 (19.0%)** | **11/21 (52.4%)** | **72.7%** | **72.7** |

Note. NA = not asked because of oversight (not marked); ND = not done because of earlier responses (marked incorrect); NR = no response (marked incorrect).

With respect to the question regarding reversing the process, three students who had correctly reunitised the number as 60 tenths 7hundredths, thought that it now was a different number. (Julian: *607 whole numbers*; Donald: *67 tenths*; Patsy: *Doesn't have a decimal point.*) However, when asked whether the value had been changed, Patsy said it hadn't.

***Regrouping.*** With respect to the open task, Table 1 shows that only half the students could attempt a solution with only 4 students successful. Of these, 3 students merely mentally "shifted" the digits to the right (partitioning reunitisation). However, Paul's protocol clearly indicates facility with the regrouping type of reunitisation.

I: *What did you do to the 10 tenths?*

P: *I put 10 tenths here* (hundredths) *and added 100 to get that* (107 h).

I: *Where did you get the 100 from?*

P: *It's hundredths* (pointing to the place name and sounding very surprised at being asked something so elementary).

Because of their problems in the reunitisng task, Roger and Dennis were not asked to complete the tasks. Roger was a very high-performing student and had done the test regrouping correctly so his interview responses were bewildering. However, this set of tasks was at the end of the interview and he was very tired. Dennis, on the other hand, had not done the test item correctly.

Inappropriate responses were categorised as *syntactic* (6t 7h; 6.0t 7h; 060t 7h), *partitioning* (600t, -h), or *naïve* regrouping (50t 17h; 50t 7h).

With respect to the closed task (60t 7h = 59t __h), 4 students who were unable to do the open task could do the closed task, indicating that the open task was a novel task for these students and that the metacognitive component rather than the conceptual component of regrouping may have been the factor inhibiting them from doing the open task. As well, 4 students were unable to do the interview item but were correct on the test item. This behaviour was attributed to the internal zero in 6.07 where regrouping involves more than two adjacent places; the regrouping required in 0.43 was prototypic in that it involved just two adjacent places.

Inappropriate responses were classified as *naïve* [1 less (t) so 1 more (h)], *equivalent* [1 less (t) so 10 more (h); 1 less (one) so 100 more (h)], or *multiplicative* [1 less (t) so 10 time more (h)]. The naïve responses took no cognisance of the equivalence (1t =10h) embedded in the regrouping process as the following protocols show.

Meg: *Well, I looked at this (60t) and thought take 1 of these and put it over there.*

Steven: *Because there's 6o tenths, 59 tenths ... you need 1 more hundredth to bring it up to 60 tenths again.*

Linda: *41 and 7, 48 hundredths.* [I: Where did you get the 41 from?] *I had to have the 7 to have the 7 hundredths and it couldn't be put in the tenths. We need 1– we needed it* (pointing to hundredths) *to be 8 ... 1 tenth missing so put it over here* (h).

Linda had scored 96.3% on the test and the earlier part of this interview indicated that she had an excellent understanding of the other numeration processes but, like Roger, she was getting tired at the end of the interview.

Two students (Chloe and Neil) wrote 59t 107h but their protocols revealed very different thinking. Both students took cognisance of equivalence but Chloe looked at the equivalence between tenths and hundredths whereas Noel looked at the equivalence between ones and hundredths, hence his 107 hundredths.

Chloe: *You have to add another tenth here* (7)*so you just put it there with the 7. This* (pointing to the 10 in 107) *is the 10 that's come from here* (1 t).

Noel: *That's 59 and you need to equal 6* (ones)*and there's 100 hundredths of it makes that 6.*

Hayley's protocol reveals that the multiplicative nature of equivalence has overridden the additive structure of regrouping, resulting in her response 59t 70h.

Hayley: *Well, 59 only leaves 1 more to go into here* (indicating hundredths). [I: 1 more what though?] *1 more, uh, 10 more because 1 of those* (indicating tenths) *equals 10 of those* (indicating hundredths) *so 10 times 7 is 70.*

## Discussion and conclusions

The interview revealed that: (a) prototypic tasks tend to promote incomplete structural knowledge as evidenced by the disparity in performance between the test item (no internal zero) and the interview item (internal zero); (b) the regrouping form of reunitisation was much more difficult than the partitioning type (60.7 = 60 t 7h); (c) even high-performing students have inappropriate structural knowledge underlying their (usually successful) performance; and (d) the same answers may be based on very different thinking. However, it is also recognised that results may have been different had the task been at the beginning of the interview when they were fresh rather that at the end when they were tired.

As Figure 1 shows, reunitising is Level 3 knowledge (structural knowledge). This knowledge encompasses ternary relationships (Halford, 1993) which are more cognitively complex than the binary and unary relationships of Levels 2 and 1 respectively. Regrouping reunitisation is more difficult than partitioning and grouping reunitisations because the decomposition within the given unit links it to additive structure (another Level 3 knowledge type).

The complexity of regrouping and the current curriculum trend promoting mental and calculator computation rather than pencil-and-paper computation raise pedagogical issues. For example, should time-consuming pencil-and-paper computation be eliminated from modern curricula?

This paper takes the stance that there is a place for all types of computation. Calculators and/or mental computation should be employed when speedy and accurate calculations are required, and pencil-and-paper computation should be employed as a vehicle for promoting understanding of mathematical principles rather than procedural proficiency. For instance, mathematical principles such as set inclusion which develops the understanding that like things only can be added or subtracted and the distributive law, underlie pencil-and-paper computation and apply across whole-number, fraction, and algebra domains. The structural knowledge embedded in computational knowledge suggests that a study of computation (not merely computational procedures for "getting an answer") should be undertaken at the upper primary or lower secondary levels.

# References

Baturo, A. R. (1999). Effective classroom teaching of decimal numeration. *South East Asian Conference of Mathematics Education, 8*, 87-96.

Baturo, A. R., Cooper, T. J. (1999). Fractions, reunitisation and the number-line representation. *International Group for the Psychology of Mathematics Education, 23*(2), 61-67.

Baturo, A., & Cooper, T. J. (1997). Reunitising hundredths: Prototypic and nonprototypic representations. *International Group for the Psychology of Mathematics Education, 21*(2), 57-64.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Behr, M., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 91-126). New York: Academy Press.

Derry, S. J. (1996). Cognitive schema theory in the constructivist debate. *Educational Psychologist, 31*(3/4), 163-174.

Halford, G. S. (1993). *Children's understanding: The development of mental models*. Hillsdale, NJ: Lawrence Erlbaum.

Heirdsfield, A. (1999). Mental addition and subtraction strategies: Two case studies. *Mathematics Education Research Group of Australasia, 22*, 253-260.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193.

Leinhardt, G. (1988). Getting to know: Tracing students' mathematical knowledge from intuition to competence. *Educational Psychologist, 23*(2), 119-144. Hillsdale, NJ: Lawrence Erlbaum.

Prawat, R. S. (1989). Promoting access to knowledge, strategy, and disposition in students: A research synthesis. *Review of Educational Research, 59*(1), 1-41.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. Journal for Research in Mathematics Education, 20(1), 8-27.

Sowder, J. T. (1988). Mental computation and number comparison: Their roles in the development of number sense and computational estimation. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades (pp. 182-197). Reston, VA: National Council of Teachers of Mathematics.

Stacey, K. & Steinle, V. (1999). A longitudinal study of children's thinking about decimal: A preliminary analysis. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 233-240).

Wearne, D., & Hiebert, J. (1988). Constructing and using meaning for mathematical symbols: The case of decimal fractions. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 220-235). Reston, VA: National Council of Teachers of Mathematics.

# FACTORS INFLUENCING TEACHERS' ENDORSEMENT OF THE CORE MATHEMATICS COURSE OF AN INTEGRATED LEARNING SYSTEM

Annette R. Baturo, Tom J. Cooper, Gillian C. Kidman and Campbell J. McRobbie
The Centre for Mathematics and Science Education
Queensland University of Technology, Brisbane, Australia

*This paper reports on a study that examined how the core mathematics course of an integrated learning system (ILS) was used to assist the teaching of mathematics in Years 1 to 10 classrooms in 23 schools in Queensland, Australia. The core mathematics activities were delivered randomly in the form of electronic worksheets; students' responses were evaluated by the ILS and successful students were progressed through the various levels. Data for the study was collected to identify operational characteristics (how the system operated in the school and individual classrooms), user characteristics (beliefs and values of the students and teachers), and teacher endorsement of the ILS. A model was developed to show the effects of operational and user characteristics on endorsement of the ILS.*

The ILS in this study is a computer based learning system comprising extensive courseware (ie. substantial course content), an aggregated learner record system, and a management system which "will update student records, interpret learner responses to the task in hand and provide performance feedback to the learner and teacher" (Underwood, Cavendish, Dowling, Fogelman, & Lawson, 1996, p. 33). It delivers courses to each student individually, manages all student enrolment and performance data, and designates which tasks have to be completed. Its management system provides the means for teachers, laboratory managers, and administrators to organise the use of courses and to monitor student progress.

This paper reports on a study to determine teachers' reactions to the ILS's core mathematics course.

***The core mathematics course.*** The core mathematics course is a closed system where curriculum content and learning sequences are not designed to be changed or added to by either the tutor or the learner (Underwood et al., 1996). It is designed to consolidate already introduced mathematics topics, not to teach new topics. The course is divided into a range of *topics* (e.g., numeration, multiplication, fractions) which are then sub-divided into collections of tasks that are sequenced in terms of performance at different *levels*. When students achieve high mastery at one level, the ILS automatically raised them to the next level. The random nature of the presentation ensures that task performance correctly represents level. The worksheets vary in quality, but many are generally attractive in their presentation and creative in the way they probe understanding, particularly with the use of 2-D representations of appropriate teaching materials in mathematics (e.g., Multi-base Arithmetic Blocks, Place Value Charts, fraction and decimal diagrams). The ILS provides online student resources: *Help* (provided answer), *Tutorial* (how to do a task), *Toolbox* (calculators, rulers, tape measures, etc.), *Reference* (definitions), and *Audio* (reads text the students through earphones). The worksheets can be printed to provide off-computer activity.

***Deficits of the ILS delivery with respect to the study.*** The random nature of the worksheet presentation means that the ILS does not provide sequences of activities that can change misconceptions. Therefore, in this study, students may "jump" from area models of fractions to number line models to set models within a short space of time. There is also a tendency for questions to be closed and to base performance on speed (with time delays resulting in the ILS defaulting to incorrect). The use of the Help and Tutorial icons automatically grades performance as incorrect thus, because of their focus on rising through the levels as rapidly as possible, the students in this study tended to avoid using these aids.

The very nature of the ILS marginalises the teacher's role and removes students' initiative and autonomy (Bottino & Furinghetti, 1996). Furthermore, the one-student-at-a-time structure means there is no place for groups. This is contrary to modern views that learning with computers should be cooperative (Sivin-Kachala, Bialo, & Langford, 1997), particularly with respect to higher cognitive functioning (Carnine, 1993; Riel, 1994), investigations and the construction of links (Wiburg, 1995). There appears to be insufficient task variety to prevent repetition so that the students in this study tended to become bored. Some tasks have novel presentation and solution formats which students found difficult to interpret (e.g., the units must be typed first). The ILS does provide feedback to students on the correctness of their responses (desirable for effective learning according to Sivin-Kachala, Bialo, & Langford). However, its worksheet nature makes it susceptible to the same pedagogical flaws as were found by Erlwanger (1975) in the Individually Prescribed Instructional (IPI) packages that proliferated in the US in the 70s (Baturo, Cooper & McRobbie, 1999).

***Learning and the ILS.*** In a re-analysis of studies into the effectiveness of the ILS, Becker (1992) found that there is very little evidence of the ILS improving student learning. He argued that the only significant improvements were found in studies supported by the manufacturers and that these had flaws. A more modern study by Underwood et al. (1996) found some statistically significant improvements from the use of the core mathematics course in primary and secondary classes, although primary numbers were too low to meet Becker's criteria for significance.

Nevertheless, the ILS is reasonably popular in many schools in Queensland, Australia. This paper explores the reasons for this and proposes a model to explain factors that appear to influence the ILS's effectiveness as perceived by teachers.

## Method

***Subjects.*** Twelve primary schools (P1 to P12 in Table 1) and eleven secondary schools (S1 to S11 in Table 1) were involved in the study. The students in most of these schools were from low socioeconomic backgrounds. The number of systems in the schools varied from one to thirty. In the larger schools, the systems were used differently in different classes. Hence, the 23 schools became 30 different cases with each case being a set of teachers, students and systems that operated with respect to the ILS in a reasonably coherent and consistent manner.

***Data collection methods.*** Two semi-structured interviews were developed for school administrators, computer coordinators, teachers, teacher-aides and technical staff (where available) for both these visits. The first interview focused on logistics and management of the ILS, beliefs about teaching and learning and the ILS's role, existing use of computers by students, and perceptions of students likes, dislikes and preferences with respect to the mathematics component of the ILS. The final interview focused on changes in logistics, management, perceptions of students' attitudes and performance, and teachers' recommendations for the ILS.

***Procedure.*** Each case was using the ILS for the first time with their students and they were visited at the start of this use (for the first interview) and then 6 months later (for the final interview). For most cases, some time was spent in classrooms observing teaching and some students were observed while working on the ILS. The interviews were audio-taped and field notes kept of observations.

***Analysis.*** The interviews were transcribed into protocols and the field notes were restructured and summarised. The data for each of the 30 cases were combined and restructured and studied for commonalities; categories (called *case characteristics*) which related to the possible impact of the ILS were identified, and responses for each case were summarised into tables. Finally, data on case characteristics was related to teachers' evaluations (endorsements) of the ILS to identify factors that appear to influence its effectiveness as perceived by teachers.

## Results

***Case characteristics and endorsement.*** As a result of the analysis of the information obtained from the school visits, the following characteristics were identified and the cases' performances on them categorised as high (H), medium (M) and low (L) as shown (for Table 1).

(1) Operational characteristics:

- *Number of computers.* H – 15 or more computers with the ILS, M – 5 to 14 computers, L – less than 5 computers;
- *System set-up (Extent computers are networked).* H – fully networked, M - peer-to-peer networked, L – not networked (stand alone);
- *Location (of computers in terms of classroom).* H – in the classroom, M – in an office, L – in a laboratory;
- *The quality of the supervision.* H - teacher monitors performance and provides remediation or students record problems for later work with teacher, M - teachers-aide supervision on ILS procedures only, L - superficial monitoring by teacher as teaches rest of class or no supervision (N).

(2) Classroom User characteristics:

- *Students' overall achievement level.* H – high achieving, M – mixed, L - low;

- *Students' computing experience.* H - creative, in that there is extensive use in a student-centred problem solving manner, M - special computer studies units, L - limited to word processing and some games;
- *The teachers' computer knowledge.* H - extensive, M - competent or limited with particular experience of the ILS, L - limited (L);
- *Degree of integration of ILS with other class work of students.* H - full integration, M - partial integration, L - no integration;
- *Extent of system of external rewards for student achievement on the ILS.* H - extensive system of rewards, M - some rewards, L - no rewards.

(3) Teacher-belief characteristics:

- *The compatibility of pedagogy with that of ILS.* H - compatible, M - pragmatic, L - incompatible;
- *The satisfaction with inservice for the ILS.* H - satisfied, M – unsure, L - not satisfied;
- *The satisfaction with the ILS's worksheet delivery.* H- satisfied, M – unsure, L - not satisfied;
- *The perception of effect of ILS.* H - positive for learning and affect, M - positive for affect only or unsure, L - negative learning and affect.

***Teacher endorsement.*** Teachers from 9 cases strongly supported the ILS (*full endorsement*), 15 cases supported with reservations (*partial endorsement*) mainly due to the cost of the software (although 3 cases had reservations about operational factors), while the remaining 6 cases did not support the ILS (*no endorsement*).

***Relation between endorsement and case characteristics.*** Table 1 provides a summary of endorsement with respect to operational characteristics, user characteristics, and teacher belief characteristics respectively. It reveals that there appear to be no differentiating operational characteristics with respect to computers that impacted on endorsement (ie. number, system, and location appear to have no relation to endorsement). However, there appears to be a weak relation to supervision in that student-based and remedial supervision seems to indicate endorsement. With respect to the user characteristics, there is no relation between endorsement and achievement rating but all other characteristics appear to be significantly related. Creative computing experience appears to lead to no endorsement as does extensive computer knowledge by teachers. Full and partial integration and external rewards appear to be related positively to endorsement. The teacher-belief characteristics show that all four characteristics (*pedagogy, inservice satisfaction, ILS delivery satisfaction, and perception*) appear to relate to endorsement.

## Discussion

In summary, the factors that appear to relate to endorsement were: the operational characteristic of supervision; the user characteristics of students' computer experience, teachers' computer knowledge, integration and rewards; and the teacher belief characteristics of pedagogy, inservice satisfaction, delivery satisfaction and effect on students. Overall, cases with the following characteristics would most likely endorse the core mathematics course of the ILS:

**Table 1**

*Endorsement With Respect to Operational, User and Teacher-Belief Characteristics.*

| Case | Operational | | | | User | | | | | Teacher belief | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | Students | | Teachers | | | | | | |
| | Number of computers | Extent of networking | Location of computers | Quality of supervision | Students' achievement | Students' comp. Experience | Teachers' comp. knowledge | Integration of ILS | Extent of rewards | Teachers' pedagogy | Teachers' sat. with inservice | Teachers' sat. with delivery | Teachers' perception |
| | | | | | | | *Full Endorsement* | | | | | | |
| P1 Y1 | L | M | H | M | M | L | L | H | L | H | H | H | H |
| Y4 | M | M | H | M | M | L | L | M | M | H | L | H | M |
| Y5 | M | M | L | H | M | L | M | H | F | H | H | H | H |
| P7 | M | L | H | M | M | L | L | M | M | H | H | H | H |
| P10 | L | L | M | H | M | L | M | M | M | H | L | L | H |
| P12 | M | M | L | H | L | L | M | H | H | H | L | H | M |
| H7 | M | L | L | M | L | M | M | M | L | H | L | H | M |
| H8 | H | H | L | M | L | M | M | M | L | M | L | H | M |
| H10 | M | H | L | H | L | M | M | L | M | H | L | H | M |
| | | | | | | | *Partial Endorsement* | | | | | | |
| P2 | L | L | H | M | L | L | M | M | M | M | L | H | H |
| P3 | L | L | H | M | M | L | L | M | L | H | L | H | H |
| P4 B | L | L | L | L | M | L | M | L | M | M | L | H | M |
| P5 | L | L | H | M | L | L | L | L | L | H | L | H | H |
| P8 | L | L | H | M | L | L | L | M | L | M | H | H | M |
| P9 | L | L | H | M | M | L | L | M | L | M | L | H | M |
| P11 | M | L | H | M | M | L | L | M | M | M | L | H | H |
| H1 | M | H | L | M | L | M | M | M | L | M | L | H | L |
| H2 | M | L | L | H | L | M | M | M | M | H | L | H | H |
| H3 | H | M | H | M | M | M | L | L | L | M | L | H | M |
| H5 A | H | H | L | H | M | M | M | M | L | H | H | H | H |
| C | H | H | L | H | M | M | L | H | L | H | H | H | H |
| H6 | M | H | L | M | L | L | M | H | M | H | H | H | H |
| H9 | M | H | L | H | L | M | M | H | M | H | H | H | M |
| H11 | L | L | H | M | L | M | L | L | L | M | L | H | M |
| | | | | | | | *No Endorsement* | | | | | | |
| P1 6 | M | M | H | M | M | H | H | L | L | L | L | L | M |
| 7 | M | M | H | M | M | H | H | L | L | L | L | L | M |
| P4 A | L | L | H | L | M | H | H | L | L | L | L | L | M |
| P6 | L | L | L | L | M | H | H | L | L | L | L | L | L |
| H4 | H | H | L | M | L | M | H | M | L | M | L | L | M |
| H5 B | H | H | L | M | M | M | L | L | L | M | H | H | M |

Note: H – high, M – medium, L - low

- teacher compatibility with the pedagogy of the ILS and teacher satisfaction with inservice, delivery of workesheets, and student learning and affect;
- remedial supervision
- integration of the ILS with the other mathematics work of the class;
- an external reward system; and
- low teacher computer knowledge and limited experience of computers by students.

***Teacher beliefs and knowledge*.** There was almost a direct relationship between endorsement and teachers' beliefs about the ILS. If teachers had concerns, particularly at the start, they tended not to endorse it. As one endorsing teacher stated: "The first thing is, it won't work if teachers do not believe in it!" Opposition to the ILS was often related to teachers' knowledge of the educational use of computers in classrooms. Teachers with extensive knowledge believed that students should not be passive with respect to computers, and tended not to useworksheet based software such as the core mathematics course of the ILS. They tended to provide their students with many creative computer experiences. As a consequence, some of their students found the ILS boring and were reluctant to continue with the program. On the other hand, teachers with limited computer knowledge liked the mathematics component of the ILS because it required little from them in terms of organising what to do with the software. If their students also had limited experience with computers, the ILS was an exciting experience with its varied displays and use of mouse and earphones.

***Instructional factors*.** The cases showed that it was important for students' mathematics experiences on the ILS to be integrated with other class work. Strong endorsement always occurred when teachers could immediately remediate ILS mathematics difficulties. If this was not possible, then endorsement appeared to be related to the level of integration of ILS activity with other mathematics teaching. This integration did not have to be undertaken by the teacher; in one strong endorsement case, the teacher organised the students to write down any mathematics errors and consult with her later. It also did not have to be content oriented; in another strong endorsement case, students found working on the ILS easy because their class rotated through workstations with individual contracts. When there was little integration, this was often when teachers used traditional expository methods, students seemed unused to rostered activities where they wereunsupervised and had to be self-motivated. This may be one reason why many of the apparently successful programs had systems of external rewards for perseverance at the computer.

In normal use, the mathematics component of the ILS treated students very passively, not a positive learning environment for many students. To overcome this, some of the most strongly endorsing teachers provided the students with a scaffold of activity and choice; allowing students to choose which mathematics activities from the ILS to do or undertake "practice sessions" on particular topics.

***Supervision***. Endorsement was strongly linked to supervision. Laboratories appeared to offer positives for supervision; whole classes could be supervised by their teachers. However, some school structures worked against this. In one case, the class teacher could not be timetabled for the ILS laboratories; in another case, students from a variety of classes were placed together in a laboratory and supervised by a teacher-aide. Classrooms with small numbers of computers had supervision difficulties; teachers had to teach the remainder of the class. However, this could be overcome if the ILS sessions were integrated into the classes' normal mathematics teaching and rostering onto the computers organised around interesting class activities and teaching of new topics. Poor supervision usually led to avoidance behaviour (mislaying passwords, guessing answers, continuously pressing the help key) and, sometimes, to unproductive and destructive actions (breaking earphones).

## Conclusions

There are three interesting conclusions from this study; the number of teachers that endorsed, the factors that effected endorsement, and the types of teachers that endorsed. In terms of numbers, a large majority of the 30 cases endorsed the core mathematics course of the ILS; they felt it had assisted their students. They wanted to recommend it to other schools (although many were concerned by its cost). But, according to the literature, the ILS does not represent software that should be effective in teaching students.

The question is, therefore, why did so many teachers endorse it?

In terms of factors, the core mathematics course of the ILS was endorsed in the cases where there was strong supervision, follow-up of students' difficulties, integration with other teaching, external rewards and some novelty with respect to computers. The chance for endorsement appeared to diminish if teachers did not support it philosophically, if rosters were inflexible and if more exciting computer options were available. The teachers who endorsed did so because they believed their class performance had improved. These factors are illustrated in Figure 1.

So what do these factors say about the number that endorsed?

The cases in this study were not randomly chosen from amongst the teaching population. They represented schools that wanted to buy the ILS. It seems, therefore, that cases that bought and endorsed the ILS had mathematics-teaching approaches and computer experiences that reflected how the core mathematics course of the ILS operated; that is, transmission models of teaching and students passively following the computer as tutor. Their students were also from poorer communities. Thus, the future of integrated learning systems such as that in this study may be problematic as teacher knowledge of educational uses of computers increases, and student experience with computers proliferates. However, transmission models of teaching have been very resilient and resistant to change and so a future for this ILS may remain.

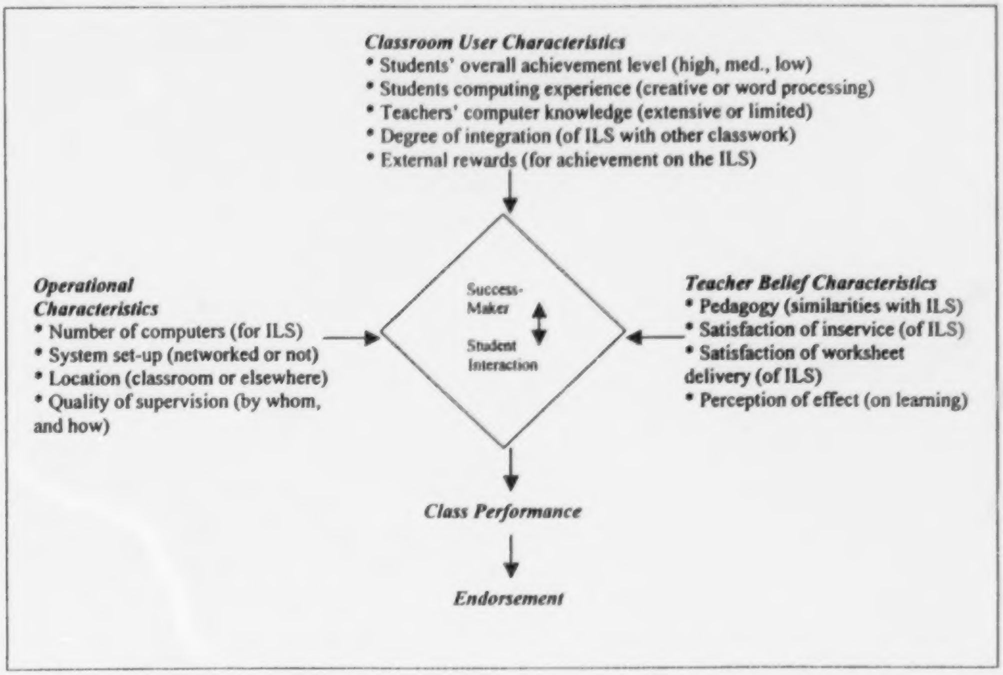

Classroom User Characteristics
• Students' overall achievement level (high, med., low)
• Students computing experience (creative or word processing)
• Teachers' computer knowledge (extensive or limited)
• Degree of integration (of ILS with other classwork)
• External rewards (for achievement on the ILS)
Operational Characteristics
• Number of computers (for ILS)
• System set-up (networked or not)
• Location (classroom or elsewhere)
• Quality of supervision (by whom, and how)
Success-Maker
Student Interaction
Teacher Belief Characteristics
• Pedagogy (similarities with ILS)
• Satisfaction of inservice (of ILS)
• Satisfaction of worksheet delivery (of ILS)
• Perception of effect (on learning)
Class Performance
Endorsement

*Figure 1.* Factors that influence the effectiveness of the ILS (Source: Modified from Baturo, Cooper & McRobbie, (1999).

## References

Baturo, A.R., Cooper, T.J., & McRobbie, C.J. (1999). Karen and Benny: Déjà vu in research. *International Group for the Psychology of Mathematics Education*, 23(2), 73-80.

Becker, H. J. (1992). Computer-based integrated learning systems in the elementary and middle grades: A critical review and synthesis of evaluation reports. *Journal of Educational Computing Research, 8*(1), 1-41.

Carnine, D.W. (1993). Effective teaching for higher cognitive functioning. *Educational Technology, 33*(10), 29-33.

Erlwanger, S. H. (1975). Case studies of children's conceptions of mathematics. *Journal of Children's Mathematical Behaviour, 1*(3), 157-283.

Riel, M. (1994). Educational change in a technology-rich environment. *Journal of Research on Computing in Education, 26*(4), 452-473.

Sivin-Kachala, J., Bialo, E.R., & Langford, J. (1997). *Report on the effectiveness of technology in schools: 1990-1997.* Washington, DC: Software Publishers Association.

Underwood, J., Cavendish, S., Dowling, S., Fogelman, K., & Lawson, T. (1996). Are integrated learning systems effective learning support tools? *Computers in Education, 26*(1), 33-40.

Wiburg, K. (1995). Becoming critical users of multimedia. *The Computing Teacher, 22*(7), 59-61.

# STUDENTS' CONCEPTIONS OF THE INTEGRAL

Jan Bezuidenhout and Alwyn Olivier

University of Stellenbosch, South Africa

*Written tests and interviews were used to investigate first-year university students' understanding of fundamental calculus concepts. The analysis of students' written and verbal responses to test items provided valuable information on the nature and characteristics of students' conceptions about key calculus concepts. In this paper we report on students' conceptions of some aspects of the integral concept. Findings of this study suggest that many of the students lacked appropriate conceptions of the integral.*

## 1. INTRODUCTION

The integral concept is characterised by an extensive internal structure, which gives the concept a multifaceted nature, and a dimension that is not reflected by a single definition, such as the Riemann definition of the integral. Because of the complex nature of the integral concept, various levels of abstraction are involved in the creation of well-developed knowledge structures of the concept. A thorough understanding of the concept necessitates appropriately constructed mental links between procedural and conceptual knowledge related to the concept. A key issue regarding calculus students' understanding of '*integral*' is the nature and quality of the conceptions they develop about the concept.

The main purpose of the research project was to gain more information regarding students' understanding of fundamental calculus concepts, after the concepts concerned had been dealt with in first-year calculus courses (Bezuidenhout, 1998a, b; Bezuidenhout, Human and Olivier, 1998). In this paper we focus on students' understanding of some procedural and conceptual aspects of the integral concept.

## 2. THEORETICAL CONSIDERATIONS

The term '*concept image*' refers to the total cognitive structure (in an individual's mind) that is associated with a specific mathematical concept (Tall and Vinner, 1981). Such a total image of a mathematical concept can be considered as the form in which the concept is accommodated in the mind of an individual. Concept images may include meaningful ideas, as well as ideas that are contrary to the meanings and formal definitions of the concepts. In some cases concept images may differ in various respects from the formal concepts as defined and accepted by the mathematical community at large. Students' misconceptions reflect such differences.

The process-object duality of mathematical concepts refers to two ways in which mathematical concepts can be conceived, because of the dual nature of such concepts (Sfard, 1991). A process conception identifies with the operational nature of a particular concept, and an object conception with its structural nature. The theory of reification described by Sfard and Linchevski

(1994) deals with the transition from a process-oriented conception to an object-oriented conception whereby a conceptual entity in its own right is created. Dubinsky (1991) describes the cognitive process of constructing a (static) object from a (dynamic) process as a form of reflective abstraction, which he calls encapsulation. Such a construction underlies an individual's ability to deal with a concept as either a process or as an object. It also underlies the ability to interpret symbolic representations of a concept both procedurally and conceptually. The formation of object conceptions from mental processes goes hand in hand with the development of mental links between procedural and conceptual aspects of mathematical concepts. An individual's process and object conceptions of a concept form part of the concept image such an individual has of that concept.

## 3. METHOD

The method used to investigate students' concept images consisted of three phases. In the first phase 107 first-year university engineering students wrote the preliminary test.

The final diagnostic test was compiled after analysis of the results obtained from the preliminary test. 523 first-year students from three South African universities participated in the final testing that was conducted near the end of the last semester of 1995. This group included students in engineering, the physical sciences and students enrolled in service calculus courses. For the analysis of test results, a random sample of 100 answer-books was taken from the three different calculus groups that participated. The composition of the sample of 100 answer-books was as follows: 35 (engineering); 35 (physical sciences); 30 (service calculus courses). The third phase involved task-based interviews with 15 students who had written the final test. For administration purposes the students were numbered $S_1$ to $S_{15}$. The interviews were structured around specific test items selected from the final test. All interviews were audiotaped.

## 4. TEST ITEMS

A selection of test items that deal with '*integral*' and some aspects related to the concept, appears in the appendix to this paper. We make some remarks concerning test items 3.3, 3.4 and 4.4 and then analyse students' responses to these items.

### 4.1 Test item 3.3

An important aspect regarding question 3 is students' interpretations of symbols that involve Riemann sums. Test item 3.3 required the student to find the limit of a specific Riemann sum. Instead of finding the limit by means of interpreting $\lim\limits_{n\to\infty} S_n(1)$ as $\int_0^1 (t^2 - 1)\,dt$, the student may opt for the procedural approach that involves the simplification of $\sum_{i=1}^{n}(f(\frac{i}{n}))\frac{1}{n}$, where $f(\frac{i}{n}) = \frac{i^2}{n^2} - 1$, so that $\frac{1}{6}(1 + \frac{1}{n})(2 + \frac{1}{n}) - 1$ is obtained. This then leads to $\lim_{n \to \infty} S_n(1) = -\frac{2}{3}$. Apart from specific procedural knowledge, this approach also requires the knowledge that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. Success with this approach is based on an appropriate process conception relating to the process that is associated with the limit of a Riemann sum as represented by $\lim_{n \to \infty} \sum_{i=1}^{n} (\frac{i^2}{n^2} - 1)\frac{1}{n}$. The construction of such a process conception is an example of a form of reflective abstraction, which Dubinsky (1991) calls interiorisation.

The approach that leads to the evaluation of $\int_0^1 (t^2 - 1)\,dt$, requires an object conception that enables the student to interpret the limit of the specific Riemann sum as the definite integral of *f* from 0 to 1. Such an object conception is indeed the conceptual entity that results from encapsulating the process associated with the limit of a Riemann sum. It is a construction that enables the student to think about '*integral*' as a limit process and, on the other hand, to treat that kind of limit process as fundamental to the integral concept. To understand the meaning of $\int_0^1 f(t)\,dt$ means to be able to go back to the process from which the object conception originates. The evaluation of $\int_0^1 (t^2 - 1)\,dt$ necessitates another construction. It is the construction of an appropriate process conception that can transform $\int_0^1 (t^2 - 1)\,dt$ into $[\frac{1}{3}t^2 - t]_0^1$. It would seem as if the conceptual route of this approach amounts to a much simpler task than the procedural route of the other approach. However, a student's options of how to deal with $\lim_{n \to \infty} S_n(1)$ depend on the space he or she is allowed to manoeuvre cognitively in this situation. The absence of relevant process and object conceptions has the effect that a student's ability to manoeuvre cognitively is restricted, so that the student may be left without any real options. On the other hand, the construction of a collection of relevant process and object conceptions creates cognitive manoeuvring space so that it becomes possible not only to apply different approaches, but also to identify the approach that best suits the situation.

### 4.2 **Test items 3.4**

Approaches and conceptions similar to those mentioned in paragraph 4.1 are applicable to the limits in test items 3.4.1 and 3.4.2. A fundamental difference between the limit in test item 3.4.2 and the limits in 3.3 and 3.4.1 is that the first one represents a function (an indefinite integral) whereas each of the other two represents a real number. The interpretation of $\lim_{n\to\infty} S_n(x)$ as a function requires an additional mental construction. If $\lim_{n\to\infty} S_n(x)$ can be perceived as representing the function process in $x \to \int_0^x f(t)\,dt$, the value of $\lim_{n\to\infty} S_n(x)$ for a particular value of $x$ can be obtained by means of integration.

### 4.3 Test item 4.4

A meaningful decision on the correct location of the local maximum of $p$ is based on an understanding of $p(x) = \int_a^x h(t)\,dt$. Such an understanding points to an appropriate object conception of the function $p$. It involves the ability to conceive of $p(x)$ as acting on $h(t)$ as a whole to produce the function process that is inherent in $x \to \int_a^x h(t)\,dt$.

## 5. STUDENTS' PROCEDURES AND CONCEPTIONS

Information that was gathered by analysing students' written and verbal responses to test items revealed various misconceptions of key calculus concepts, as well as typical errors arising from the application of such misconceptions. In this section the emphasis is on some erroneous procedures and misconceptions pertaining to students' thinking of the integral concept and symbols related to it.

### 5.1 Procedures and conceptions: test items of question 3

The 30 calculus course students' answers were not taken into account for the test items of question 3, because they, in general, did not receive formal instruction in the interpretation and use of Riemann sums.

Incorrect answers include 0 (23 students) and $\frac{2}{3}$ (5 students). The following is an example of a procedure that resulted in 0 as the answer:

$$\lim_{n\to\infty} S_n(1) = \lim_{n\to\infty} \left(f(\tfrac{1}{n})\tfrac{1}{n} + f(\tfrac{2}{n})\tfrac{1}{n} + \ldots + f(1)\tfrac{1}{n}\right) = 0$$

Sixteen of the students who indicated that the answer is 0, did not present any written procedure. During an interview session with one of those students the following procedure was produced in support of his answer:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(\tfrac{i}{n}) \times \tfrac{1}{n} &= \lim_{n\to\infty} \left[f(\tfrac{1}{n}) \times \tfrac{1}{n} + f(\tfrac{2}{n}) \times \tfrac{1}{n} + \ldots + f(\tfrac{n}{n}) \times \tfrac{1}{n}\right] \\
&= \lim_{n\to\infty} \left[(\tfrac{1}{n^2} - 1)\tfrac{1}{n} + ((\tfrac{2}{n})^2 - 1)\tfrac{1}{n} + \ldots + ((\tfrac{n}{n})^2 - \tfrac{1}{n})\tfrac{1}{n}\right] \\
&= 0
\end{aligned}$$

Following a question of the interviewer on how 0 in the last step of the procedure was obtained from the preceding one, the student claimed that

$$" \lim_{n\to\infty}\left[\left(\tfrac{1}{n^2}-1\right)\tfrac{1}{n}+\left(\left(\tfrac{2}{n}\right)^2-1\right)\tfrac{1}{n}+\ldots+\left(\left(\tfrac{n}{n}\right)^2-\tfrac{1}{n}\right)\tfrac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left(\tfrac{1}{n^2}-1\right)\tfrac{1}{n}+\lim_{n\to\infty}\left(\left(\tfrac{2}{n}\right)^2-1\right)\tfrac{1}{n}+\ldots+\lim_{n\to\infty}\left(\left(\tfrac{n}{n}\right)^2-\tfrac{1}{n}\right)\tfrac{1}{n}"$$

from which he deduced that the answer is 0. The student based this last-mentioned part of his procedure on the limit law "*the limit of a sum is the sum of the limits*". Such a use of that limit law to deal with the "*limit of a Riemann sum*" points to the evocation and application of an inappropriate process conception. It is quite possible that some other students that also indicated 0 as the answer, applied similar process conceptions. Results of this test item indicate that many students lacked appropriate conceptions for dealing with $\lim_{n\to\infty}\sum_{i=1}^{n} f(\tfrac{i}{n})\tfrac{1}{n}$.

The following erroneous procedure is an example of one that results in $\tfrac{2}{3}$ as an answer: $\int_0^1 (t^2-1)\,dt = \left[\tfrac{t^3}{3}-t\right]_0^1 = \left|\tfrac{1}{3}-1\right| = \left|-\tfrac{2}{3}\right| = \tfrac{2}{3}$. Interviews with students revealed that an object conception that provides for only a positive value of the definite integral, inasmuch as it connects the integral with the area between the curve and the $t$-axis (horizontal axis), is a source of such a procedure.

The most frequent incorrect answer for test items 3.4.1 and 3.4.2 was 0. For each of the two test items 11 students arrived at 0 as the answer. The following excerpt from a written explanation of one of the students points to a way of thinking about a symbolic expression like $\lim_{n\to\infty}\sum_{i=1}^{n} f(\tfrac{i}{n})\tfrac{1}{n}$ that probably underlies many of the procedures that resulted in 0 as an answer (test items 3.3, 3.4.1 and 3.4.2 are concerned): " $\lim_{n\to\infty} S_n(1) = 0$ *because of* $\tfrac{1}{n}$ *which has to be multiplied in each case ...* $\tfrac{1}{n} \to 0$ *as* $n \to \infty$*. In a similar fashion ...* $S_n(3)$ *will also tend to zero because of the presence of* $\tfrac{1}{n}$ *...Because we know that* $S_n(1)$ *and* $S_n(3)$ *both tend to zero as* $n \to \infty$*, and it is seen that* $\tfrac{1}{n}$ *is always present even in the nth term, we can say that* $\lim_{n\to\infty} S_n(x) = 0$". Because of the way in which this explanation dwells on "*the presence of* $\tfrac{1}{n}$" one senses the erroneous '*limit-of-a-sum-is-the-sum-of-the-limits*' interpretation that was mentioned in the discussion above.

### 5.2 Procedures and conceptions: test item 4.4

The distribution of the 100 students' answers to test item 4.4 is as follows: correct (16 students); incorrect (73 students); no answer (11 students). Forty-eight of the students indicated d as the answer to test item 4.4 and another 7 gave an answer that included d. This incorrect answer, namely d, was the most common one. It would appear that many students confused the function *h* with the function $p$. A student who had indicated a local maximum at $d$ mentioned during an interview session that, "*for a maximum value one has to consider the derivative, or the gradient*". The student also referred to a horizontal tangent to the graph at $d$.

Students' explanations during interviews have revealed reasons for some of the incorrect answers for this test item. One kind of unsatisfactory interpretation seems to result from the application of a conception where '*integral*' is perceived as the '*area between the graph and the horizontal axis*'. It is a conception that involves the '*idea*' that a definite integral such as $\int_a^b h(t)\,dt$, with $a < b$, represents an area even if $h(t) < 0$ on $[a, b]$, as is the case in test item 4.4. Such an inappropriate '*area-conception*' of integral stems from the special case, $f(x) \geq 0$, as its construction is based on a generalisation of the special case. It is clear that the application of such a conception may result in satisfactory answers in some mathematical situations (for example where $h(t) \geq 0$), but in others it may certainly produce unsatisfactory results.

Student $S_5$ was one of the students that demonstrated an '*area-conception*' as described above. He thought of the function $p$ as representing "*the area between the graph, or the function, and the t-axis*" (the graph he is referring to is that of $h$). It is on the basis of this interpretation of $p$ that student $S_5$ reckoned that the function values of $p$ increase as $x$ increases on the interval $[a, g]$: "*$p(x)$ will increase … it will increase up to $g$*". Student $S_5$ indicated $p(g)$ as the maximum value of $p$ in the interval $[a, g]$, "*because it gives the total area*". Due to his interpretation of the function $p$ the student arrived at the answer $g$.

Student $S_6$ mentioned that "*the local maximum of $p(x)$ is at $c$*". He described the function $p$ as "*increasing from a to c*", because "*here* (student points at the interval $[a, c]$) *the area gets larger, larger and larger*". Furthermore, he mentioned that the function $p$ decreases on the interval $[c, e]$. In a lengthy explanation student $S_6$ conveyed the idea that, since the graph of $h$ is above the $t$-axis on the interval $[c, e]$, one has to subtract the area of the region that is bounded by the graph of $h$ and the $t$-axis on $[c, e]$, or on any subinterval $[c, x]$ of $[c, e]$, from the area of the region that is bounded on the interval $[a, c]$. For example, the student's explanation of how to find $\int_a^x h(t)\,dt$, when $x = d$, corresponds with the following procedure: $\left|\int_a^c h(t)\,dt\right| - \int_c^d h(t)\,dt$. It seems clear that the conception that was evoked here for dealing with $\int_a^x h(t)\,dt$, is in conflict with the formal definition of the definite integral.

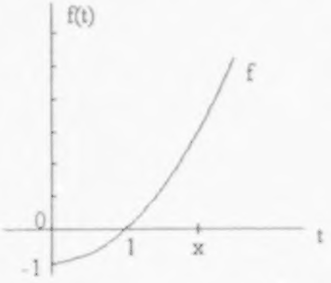
## 6. CONCLUSIONS

The research described in this paper has appeared to be effective in revealing some inappropriate conceptions forming part of university students' concept images of the integral concept. It was indicated in paragraph 5.2 by means of two examples that students' concept images may sometimes include unsatisfactory '*area-conceptions*' of integral. Those conceptions are usually due to insufficient abstraction of concept images of integral, tied to area as context, the context in which the conceptions were originally formed because of the specific teaching strategy that was followed. Concept images with such a deficiency need further abstractions and reconstructions in order to deal successfully with an integral like the one in test item 4.4. The results of test item 4.4 suggest that many students had inappropriate conceptions relating to $\int_a^x h(t)\,dt$.

The information that was gained regarding students' conceptions of procedural and conceptual aspects of the integral concept, contributed to an understanding of the nature and possible origins of such conceptions. Understanding the nature of students' conceptions is an important first step in the development of well-planned pedagogical approaches that may provide students with opportunities to construct powerful concept images.

## APPENDIX

### QUESTION 3



The curve in the figure above is that of a function $f$ defined by $f(t) = t^2 - 1$ for $t \geq 0$. Assume that the interval $[0, x]$ is divided into $n$ equal subintervals, each with length $\frac{x}{n}$, so that a partition with partition points $0, \frac{x}{n}, \frac{2x}{n}, \frac{3x}{n}, \ldots, \frac{(n-1)x}{n}, x$ is thus obtained for the interval $[0, x]$. Define $S_n(x)$ by

$$S_n(x) = (f(\tfrac{x}{n})) \times \tfrac{x}{n} + (f(\tfrac{2x}{n})) \times \tfrac{x}{n} + (f(\tfrac{3x}{n})) \times \tfrac{x}{n} + \ldots + (f(\tfrac{(n-1)x}{n})) \times \tfrac{x}{n} + (f(x)) \times \tfrac{x}{n}.$$

That is, $S_n(x) = \sum_{i=1}^{n} (f(\frac{ix}{n}))\frac{x}{n}$

3.1. Determine the sum $S_2(1)$; in other words, determine the value of $S_n(x)$ for $n = 2$ and $x = 1$.

3.2. Assume it is calculated that $S_{10000}(2) = 0{,}667$ (rounded off to 3 decimal places). Which inferences concerning the function $f$ or its graphical representation can be made on account of this information?

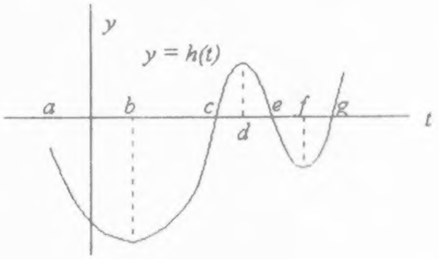
3.3. What is the value of $\lim\limits_{n \to \infty} S_n(1)$?

3.4. Determine:

3.4.1. $\lim\limits_{n \to \infty} (S_n(3) - S_n(1))$

3.4.2. $\lim\limits_{n \to \infty} S_n(x)$

## QUESTION 4



The figure above shows the graph of a continuous function $y = h(t)$ for $a \le t \le g$. Use the graph of the function $h$ to answer the following questions:

4.4. At which point(s) in the interval $[a, g]$ will the function $p$ defined by $p(x) = \int_a^x h(t)dt$ have a local maximum?

### REFERENCES

Bezuidenhout, J. (1998a). First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology*, 29(3), pp. 389 - 399.

Bezuidenhout, J. (1998b). *Universiteitstudente se begryping van basiese begrippe in elementêre differensiaal- en integraalrekening* [University students' understanding of fundamental calculus concepts]. Unpublished Ph.D. dissertation. University of Stellenbosch.

Bezuidenhout, J., Human, P. and Olivier, A. (19 [illegible]. Some misconceptions underlying first-year students understanding of "*average rate*" and of "*average value*". *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, 2, pp. 96 - 103.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking*, pp. 95 - 123. Dordrecht, Kluwer.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, pp. 1 - 36.

Sfard, A. and Linchevski, L. (1994). The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics*, 26, pp. 191 - 228.

Tall, D.O. and Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, pp.151 - 169.

# THE USE OF MENTAL IMAGERY IN MENTAL CALCULATION

**Chris Bills and Eddie Gray**
Mathematics Education Research Centre, University of Warwick
COVENTRY CV4 7AL, UK.
email: chris.bills@warwick.ac.uk and serah@warwick.ac.uk

*When asked to perform a mental calculation and then to say what was in their head when doing it, some young pupils describe visual images whilst others say it was just words. The results from the first year of a longitudinal study of the mental representations used by children aged 7 to 9 years suggest that some have a preference for mental visual images of concrete objects whilst others have images that are more abstract. There is a relation between the numbers of images of each category that individuals use in arithmetic and in non-arithmetic contexts but neither the quantity, or quality, of the pupil's images correlates with their accuracy in the mental calculations or with their achievement in written mathematics tests.*

## INTRODUCTION

Preliminary studies conducted in 1997-8 with children aged 6 to 7 years in an English school revealed that mental visual images were not often mentioned when children were asked how they had performed mental calculation (Bills, In press). The language they used, however, suggested that they had been influenced by their classroom experiences and that their mental representations were "image-like" in representing those experiences (Bills & Gray, 1999).

The longitudinal study which followed sought to explore the individual differences in these mental representations. The results reported here focus on the quantity and quality of mental visual images described by pupils after a mental calculation and in response to questions requiring a description of mental representations for procedures and concepts. The four categories of visual image defined in this paper attempt to distinguish between representations that vary from concrete/iconic to symbolic/abstract. Using these classifications the children are compared and descriptions are given of their qualitatively different styles of visual imagery. It had been conjectured that pupils' mental representation might be transformed from a quasi-sensory image (visual, auditory, tactile) to a more general, language-like, representation over different time periods. No evidence of this development in individuals was found but more pupils described abstract images at the end of the year than at the beginning. Pupils tended to have a similar quantity and quality of mental visualisations in calculation and non-calculation contexts

The literature on mental representation is extensive. The brief selection given here provides a theoretical background for the qualitative classification of visual images and for the analysis of individual's quantitative levels of visualisation. The discussion suggests that this study is in broad agreement with previous studies which show individual preferences for visual or verbal representations, though none of the other studies has explicitly compared young children's visual imagery for calculation.

## MENTAL REPRESENTATION

Paivio (1986) notes that dictionary definitions of "representations" indicate that they can be physical or mental, that they are symbolic and they vary in abstractness. In his view,(Paivio, 1971, p18) "it is simply asserting a truism" to say that modes of mental representation evolve in the individual from the more concrete to the more abstract. In comparing the theories of Bruner and Piaget, Bruner's view of this development is characterised by Paivio as a sequential emergence of three modes of representation: enactive (motor), iconic (imagery) and symbolic (verbal). Pavio suggests that Bruner ignored the symbolic nature of images . Piaget, according to Paivio , also saw the developing abstractness of symbols in terms of the development from the use of visual images, for concrete objects, to words for concepts. Piaget however regarded images and words as having complementary symbolic functions with images used for thinking about concrete features of the perceptual world. In Paivio's own view images and words constitute two interacting symbolic systems, which he refers to as a "dual code approach" (Paivio, 1971; 1986).

Piaget & Inhelder (1971) stated that in fields where evolution related to age exists, the description of stages provides a natural classification but the stages of development of images, "if there are any" (p1) is much less obvious. Piaget thus felt the need to classify different types of images and defined two broad groups: "reproductive", evoking objects and events already known and "anticipatory", representing events that have not been perceived. They can be categorised as "static" (imagined objects), "kinetic" (imagined movements) or "transformations" (imagined transformations of objects). He suggested that mental images constitute a system of "intermediary agents" between perception and general concept and that the use of images was an instance of the individual's preference for the representation of concrete instances, to give abstract concepts "exemplarity". Further more, the image, like language, serves as a symbolic instrument to allow thought to be expressed to ones self or others. Piaget regarded it as "self-evident" that individuals will concretise the words they use with personal images and that language alone can not conserve perceptual experiences in memory.

Richardson (1980) provided a philosophical discussion of the concept of mental imagery and concluded that mental imagery *is* an empirical phenomenon which *is* open to scientific investigation. In his view mental imagery constitutes a non-verbal, short-term, working memory in which information may be pictorially represented and spatially transformed. He did not consider aspects of imagery which can be approached experimentally, such as duration or vividness, to have any general theoretical significance. He suggested that the more interesting questions for cognitive psychology concern the possibility of correlations between the phenomenal experience of mental imagery and performance in cognitive tasks. He noted, however, that whilst, under laboratory conditions, instruction to use mental imagery as an aid to memory can lead to substantial improvement in memory performance, the use of mental imagery may only be of limited use in assisting learning in everyday life.

Richardson also concluded that experimental studies into individual differences in ability to use mental imagery, and preference for using it, had not been conclusive. Vividness of imagery, for instance, seems unrelated to accuracy when using mental imagery as a memory aid. There is, however, a significant positive correlation between vividness of imagery and the subject's need for approval. In general, results are affected by subjects' dispositions to behave in a socially desirable manner, i.e. to guess what is expected and to provide the researcher with what they think he wants to hear. The performance can also be manipulated by cues the researcher may give about the desired outcome.

Dehaene (1993) reviewed experimental research in numerical cognition, which explores mental processes used in number comprehension, production and calculation, and noted the prevailing notion that numerical abilities derive from linguistic competence. He gathered evidence to suggest, however, that quantification and approximation may not be reduced to symbolic or linguistic processing but instead rely on a magnitude analogue which is a separate mental representation. He thus proposed a triple code model for numerical cognition which comprises interconnected number representations: Visual (Arabic number form), Auditory/Verbal and the Analogue Magnitude Representation. He also cited research which demonstrates that use of calculation strategies does not follow a strict developmental sequence, rather that individual children switch strategies from trial to trial and that the strategy selected depends on the reliability and speed of the available strategies. He suggested that mental manipulation of a spatial image of the operation in arabic form is likely to be preferred for multi-digit calculation whilst addition and multiplication bonds are associated with the verbal word frame.

Krutetski (1976) attempted to identify verbal-logical and visual-pictorial components of able pupils' mental activity. He distinguished three types of giftedness: "analytic" (pupils who favour verbal-logical thinking even when problem suggests visual concepts), "geometric" (pupils who prefer to interpret abstract mathematical relationships visually and have difficulty reasoning without visual supports) and harmonic (pupils who show an equilibrium of verbal-logical and visual-pictorial thinking). Of 34 gifted pupils he described 6 as analytic, 5 as geometric, 13 as "abstract-harmonic" (those who can use visual-pictorial thinking but feel no need to) and 10 pupils were "pictorial-harmonic" ( those who use visual images to simplify a solution though can manage without). He noted, however, that boundaries between types are not entirely clear-cut. Low levels of visualisation amongst able students has also been reported by Presmeg (1986). In her sample of 277 final year senior high school students the 7 identified as having outstanding abilities were almost always non-visualisers. Of the 27 who were 'very good' only five were visualisers.

Presmeg's subsequent international study of Grade 11 students (Presmeg, 1995) showed that visuality scores (a measure of preference for visual methods) were normally distributed. Specht & Martin (1998) adapted Paivio's Individual Difference Questionnaire to produce a measure of imaginal and verbal thinking habits of children. With their sample of 214, 11-12 year old children, scores on the verbal items showed a normal distribution but the scores on imaginal items were negatively skewed. They found few 'pure types' (1 s.d. above the mean on one and 1 s.d. below the mean on other). Just three were pure verbalisers and 15 pure visualisers.

Literature on mental representation, and in particular on visual imagery, generally supports a view of variability within and between individuals rather than the existence of developmental levels.

## METHOD

Lesson observations and pupil interviews were conducted with two classes from Year 3 (pupils aged 7 and 8 years) in a school for children aged 5 to 11 years in a large middle-income village near Birmingham U.K., from September 1998 to July 1999. The 80 children in the year had been placed in one of three sets for Mathematics based on their previous attainments. Lessons with Set 1 (33 pupils) and Set 2 (31 pupils) were observed and a sample of fourteen pupils from each set was interviewed in December, March and July. The samples were chosen to represent the spread of achievement levels in each group and to have equal numbers of pupils who had mentioned some visual images in preliminary studies and pupils who had mentioned none. There were equal numbers of boys and girls.

Over the three interviews 40 questions were used, which were classified into 8 calculation and 6 non-calculation types, presented vebally. There was a follow up question "What was in your head when you were thinking of that?" after each question and finally "Anything to see?" if no image was mentioned by the pupil.

| Type | Description | Questions |
|---|---|---|
| 1 | 1 digit addend | 17 + 8, 17 + 9 |
| 2 | Missing addend | 13 + * = 18, 30 + * = 80 |
| 3 | 2 digit addition | 48 + 23 |
| 4 | Addition of multiple of 10 | 97 + 10, 597 + 10, 200 more than 4360 |
| 5 | Counting | What comes before 380, 2380 |
| 6 | Rounding | Round 246, 2462 to the nearest ten |
| 7 | Recent topic | What is difference 27 and 65, Read this number (26,365) |
| 8 | Recent topic | Estimate length, Read time (11:40), quarter of 40 |
| 9 | Numerical procedure | Tell me how to add 23, find a third, times by ten, add 23 |
| 10 | Non-numerical procedure | Tell me how to cross road, draw picture, do subtraction |
| 11 | Mathematical first image | First thing in head when I say centimetre, three, million |
| 12 | Maths concept image | What else can you tell me about centimetre, three, million |
| 13 | Non-Maths first image | shadow, ball, adjective |
| 14 | Non-Maths concept image | shadow, ball, adjective |

## RESULTS

### Categories of mental visual images

A total of 21 calculation questions evoked 134 mental visualisations by pupils (mean 6.4 per question). The 19 non-calculation questions evoked 103 mental visualisations by pupils (an average of 5.7 per question). The 27 pupils who took part in all three interviews had a total of 237 visualisations out of 974 responses where they gave a reply (from a total of 1080 possible) i.e. 24% of attempted answers involved some mental visualisation. Each pupil had a total of 40 questions. The mean number of visual images per pupil was 8.8 (st dev 6.1), the mean number of visualisations per pupil for calculation was 5 (st dev 4.1), two thirds of pupils had 5 or fewer images when making these 21 mental calculation, and for the 19 non-calculation questions the mean was 3.8 per pupil (st dev 2.5) .

The categories of images were defined to reflect the progression of the teacher's representations from concrete to symbolic. The distributions of pupils images in each category were:

| Categ | Classification - All images | | Calculation image | | Non-Calc image | |
|---|---|---|---|---|---|---|
| 1 | Concrete | 14% | Objects | 5% | Episodic | 26% |
| 2 | Concrete Representation | 36% | Number line, Dienes blocks | 37% | Real life | 35% |
| 3 | Symbolic Representation | 12% | Columns | 10% | Picture | 14% |
| 4 | Symbolic | 38% | Horizontal sum, Numbers | 48% | Symbol | 25% |

The frequency of each category of image and a typical example for each question type was as follows:

| Category | Type | Freq. | Calculation Visualisation |
|---|---|---|---|
| 1 | 1 | 1 | me putting some of the numbers like, just little dots |
| | 2 | 1 | I could see in my head like my fingers moving about |
| | 4 | 2 | ten frogs and I added them on |
| | 5 | 1 | I got fingers in my head |
| | 8 | 1 | I imagined he (Father) was pointing to things |
| 2 | 1 | 8 | lots of cubes in a big long line |
| | 2 | 1 | all the tens then I just counted them 10, 20, 30, 40, all that. |
| | 3 | 6 | a number line and Mrs I was holding a ruler on the board |
| | 4 | 4 | I had cubes this time. |
| | 5 | 10 | I got a picture, of like a number chart like 1, 2, 3, 4, 5, 6, 7, |
| | 6 | 6 | It was in words. Yeh cause she had written it on the board. |
| | 7 | 4 | tens, um, they're just long, uh, lines I was counting the tens. |
| | 8 | 11 | I saw Miss P moving them across, I just saw like sticks |
| 3 | 1 | 2 | tens and units It's just like in mid air (pictured columns) |
| | 2 | 1 | I seen that tens and units thing |
| | 3 | 8 | written down as a sum like (points in air to indicate vertical) |
| | 4 | 2 | Yeh a 3 in the hundred column. |
| | 7 | 1 | It was something I could see in my head, the tens and units |
| 4 | 1 | 11 | little picture of a sum. Like 17 there, 9 there, add there, equals |
| | 2 | 4 | written like a sum |
| | 3 | 10 | Some of the numbers just appeared in my head |
| | 4 | 14 | I saw a few numbers. um, hundred and a 7 Bubble writing |
| | 5 | 8 | Picture, I normally picture the number in my head, |
| | 6 | 10 | the number, apart from I keep losing it. It kind of like fades |
| | 7 | 2 | um see the number that I think it is in my head and I just say it |
| | 8 | 5 | halves, quarters wholes, um, . like 1 over 2 |

| Category | Type | Freq. | Non-Calculation Visualisation |
|---|---|---|---|
| 1 | 10 | 1 | the road with the cars on and then me pressing a button |
| | 11 | 3 | I could see, at Gala day, cause we had these kind of like bugs, |
| | 12 | 2 | All I can see is Miss P holding up a long rulers |
| | 13 | 8 | me standing in the sun and looking at the shadow |
| | 14 | 3 | me and Grandpa playing throwing it to each other |
| 2 | 9 | 3 | Well three cakes |
| | 10 | 11 | when the cars're all gone they walk across, holding hands |
| | 11 | 9 | some one holding up the number 1 thou, um, a million |
| | 12 | 1 | A ruler, the ruler that we have in our thing, |
| | 13 | 8 | I saw a big bouncing ball. |
| | 14 | 4 | picture of someone playing football, probably David Beckham |
| 3 | 10 | 1 | it was like a little person crossing the road |
| | 11 | 2 | seeing a sum, and it had a million in it, like thousands, and tens |
| | 12 | 7 | I'm seeing a person kicking a ball. A football player |
| | 14 | 4 | I like have a big picture of a ball, and imagine sort of feeling it |
| 4 | 9 | 5 | when I said 2 times 10 I had a picture of 2 times ten |
| | 10 | 1 | an arrow comes out of the traffic light and points at the crossing |
| | 11 | 12 | thirds, thirty, 300. Sort of seeing them. |
| | 12 | 2 | three threes are 9 I had the sum in my head |
| | 13 | 4 | I saw some other words. Um, . , um mad, and, . and bannanas |
| | 14 | 1 | Well I was seeing my sentence |

## Correlations

Paired correlation coefficients were calculated and t-values computed. The critical value of $t = 2.787$, for the .005 level, one tailed test of significance, reveals that correlation coefficients over $r = 0.49$ are significant. There was no significant linear relation between the following variables for each pupil:

| | |
|---|---|
| Number of Visualisation / Year 3 Written Mathematics assessment score | $r = 0.08$ |
| Number of Visualisation / Year 3 Mental Mathematics assessment score | $r = -0.12$ |
| Number of Visualisation / Year 3 Reading assessment score | $r = -0.18$ |
| Number of Visualisation / Accuracy in interview calculations | $r = 0.02$ |

Similarly there was no significant correlation when numbers of visualisations of each separate category were paired with these variables nor when the numbers of visualisations of each category used in calculation questions were paired with them.

Grouping the two concrete categories together and the two abstract categories together gives a coarser classification. Pairing the combined visualisation category 1/2 and category 3/4 with each of the measures of pupil achievement again revealed no significant correlations between them. The correlation between the numbers of Calculation and Non-calculation visualisations however showed that pupils operate with similar categories of images in calculation and non-calculation contexts:

| | |
|---|---|
| Calculation Images Category 1/2 / Non-Calc Images Category 1/2 | $r = 0.53$ |
| Calculation Images Category 3/4 / Non-Calc Images Category 3/4 | $r = 0.66$ |

## Individual differences

Pupils performance has been rated as follows:

| | |
|---|---|
| High | greater than one standard deviation above the mean |

| | |
|---|---|
| Moderate | within one standard deviation above the mean |
| Low | less than one standard deviation above the mean |

Three pupils were rated High for accuracy in interview calculations. Of these only one, Clara, was a High visualiser the other two being Moderate visualisers. The three Low accuracy pupils were all Moderate visualisers.

Six pupils were Low achievers in the Year 3 Written Mathematics test and all but one were Moderate visualisers, the exception being a low visualiser. Of the five High achievers three were moderate visualisers, one was a Low and the other a High visualisers.

Five pupils rated as High visualisers and four of these were moderate in both Accuracy in interview calculation and Year 3 Written Mathematics scores, the other, Clara, being a high achiever. The three Low visualisers were all Moderate Achievers.

There was a marked difference in visualisation styles amongst the five High visualisers. The highest, Elspeth, reported 21 images, 11 of which were category 1/2 and 10 were category 3/4. Two of the other pupils had similar proportions of each category. Mandy and Dennis however had contrasting proportions. Mandy had 17 category 1/2 images and only 2 of category 3/4 images, whilst Dennis only had 2 category 1/2 and 16 category 3/4. All 11 of Dennis' calculation images were of category 4 and 13 of Mandy's 14 calculation images were of category 2 (exclusively number lines). These two also showed a higher proportion of calculation to non-calculation images whereas the other three High visualisers had numbers of calculation and non-calculation images in proportion to the number of questions in each group.

### Development of images

Where individual pupils had images for questions of the same type in more than one interview there was no evidence of a development from a low category to a higher one. Pupils were as likely to use a lower category than previously as they were to use a higher one. There was evidence, however, of an overall increase in more abstract images in three questions (single digit addition, two digit addition and addition of 10) but due to pupils evoking abstract category visualisations in later interviews who had not had any visualisation in earlier interviews. This suggests that pupils do not necessarily develop from a low level to high level mental visualisation.

### Group Differences

Of the five High visualisers all were from Set 1 and four of them were girls. Thus initial comparisons suggested that Set 1 pupils have more visual images than Set 2 pupils and that girls have more images than boys. Independent-samples t-tests on means, however revealed no significant differences between the groups.

## DISCUSSION

In line with Presmeg's (1986) and Krutetski's (1976) findings high achievers in this study tend not to be high visualisers. The same, however, can also be said of low achievers. The proportions of visual, harmonic and verbal types of pupils are broadly the same in this study as the proportions in Krutetski's "gifted" sample.

There is no evidence from this study to support a developmental model for visual images. Teachers' representations follow a development from counting of concrete objects via concrete representations, such as Deines blocks, to symbolic representations and symbol manipulation. Pupils in this study generally have mental visualisations which span all these catagories. This is in accord with Dehaenes conclusion that strategy use can vary from trial to trial. In this study many pupils varied their category of visual image. There is also no evidence that having visual images is any assistance with accuracy or that there is any association between mathematical achievement and having mental visualisations.

The children in this study may have behaved in a "socially desirable" way and have responded to what they thought was required of them (Richardson, 1980). The true level of image use for mental calculation may thus be lower than reported. There is, however marked individual differences, which are apparent in the High visualisers mentioned above, but also in Moderate visualisers who may have a preponderence of images in one category. What is apparent is that these children have shown a preference for concrete or abstract visual images in both calculation and non-calculation contexts.

## REFERENCES

Bills, C.J. (In press). The Influence of Teachers' Presentations on Pupils' Mental Representations. *Annual Proceedings of the British Society for Research into Learning Mathematics.*

Bills, C.J., & Gray, E.M. (1999). Pupils' Images of Teachers' Representations, *23rd Conference of the International Group for the Psychology of Mathematics Education*, Vol 2 (pp 113-120) Haifa, Isreal: Programme Committeee

Dehaene, S. (1993). Varieties of Numerical Ability. In S. Dehaene (Ed.), *Numerical Cognition* (pp. 1-42). Cambridge, MA: Blackwell.

Krutetski, V.A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren* (Teller, J. Trans.). Chicago: University of Chicago Press.

Paivio, A. (1971). *Imagery and Verbal Processes.* New York: Holt, Rinehart and Winston.

Paivio, A. (1986). *Mental Representations: A Dual Coding Approach.* Oxford: Oxford University Press.

Piaget, J., & Inhelder, B. (1971). *Mental Imagery in the Child.* London: Routledge and Kegan Paul.

Presmeg, N.C. (1986). Visualisation and Mental Giftedness. *Educational Studies in Mathematics, 17*(3), 297-311.

Presmeg, N.C. (1995). Preference for Visual Methods: An International Study. M. L & C. D (Eds.), *18th International Conference for the Psychology of Mathematics Education*, Vol. 3 (pp. 58-65). Recife, Brazil: Programme Committee.

Richardson, J.T.E. (1980). *Mental Imagery and Human Memory.* London: Macmillan.

Specht, J.A., & Martin, J. (1998). Individual Differences in Imaginal and Verbal Thinking Habits of Grade 6 Students. *Journal of Mental Imagery, 22*(3&4), 229-236.

BEST COPY AVAILABLE

# READINESS FOR ALGEBRA[1]

G.M. Boulton-Lewis, T.J. Cooper, B. Atweh, H. Pillay, and L. Wilss
Faculty of Education, Queensland University of Technology

*Algebraic knowledge stems from integration of arithmetic and pre-algebraic knowledge. Current studies show this is a complex process. This paper reports on a longitudinal study designed to investigate students' readiness for algebra, in terms of prerequisite knowledge, from a cognitive perspective. In so doing it was necessary to explicate what constituted a pre-algebraic level of understanding. Thirty-three students in grades 7, 8, and 9 participated. A model for the transition from arithmetic to pre-algebra to algebra is proposed and students' understanding of relevant knowledge is discussed. Results showed inadequacies in students' prerequisite arithmetic knowledge and revealed aspects that require greater attention in early algebraic instruction.*

## Development of Algebraic Concepts

We have proposed a model (see Figure 1) that depicts the move from arithmetic to algebra as a sequence of development of knowledge that incorporates pre-algebraic knowledge. Inherent in the model are cognitive implications from Biggs and Collis' (1982) sequential development of algebraic concepts and other findings in the literature pertaining to the development of algebraic knowledge. Particular attention was paid to the fact that knowledge is cumulative and that each stage is prerequisite to subsequent stages. Additionally, we recognise that knowledge is recursive and earlier stages can be revisited when necessary. We exemplify this by analysing the knowledge needed to solve an equation, such as $x+3=2x+1$. Firstly a student must recognise symbols that represent an equation including unknowns. Secondly, the student has to determine whether to use an inverse or balance approach in handling the equals sign and to identify the sequence of operations. Thirdly, operations must be carried out correctly to determine the value of the variable. The difficulty for students is that they have to integrate knowledge of (a) symbols, numbers, and variables (b) basic computations (c) arithmetic laws for individual operations and sequences of operations (d) meaning of equals, and (e) operations on variables. Therefore we suggest that solution is dependent upon a hierarchical and recursive sequence of mathematical concepts, as depicted in Figure 1, and explained as follows.

Initially students must possess *arithmetic* knowledge including: operational laws such as inverse, commutative, and distributive; be able to apply inverse numerical procedures; and recognise equals as each side of the '=' sign being the same value. While we acknowledge that instructional practices include '=' as denoting the answer, we believe this on its own constitutes an inappropriate conception of equals. Following this we propose a *pre-algebraic* (Filloy & Rojano, 1989) level that includes: concatenation; unknown and variable; a focus on understanding equals to mean each side of an equation is the same value; using inverse procedures to solve a linear equation with an unknown; and an understanding of expressions as compared to equations. Finally knowledge of *algebraic* equation solution methods, incorporating operating on and with the unknown using balance procedures, are essential along with an understanding of equals as equivalence, that is denoting a relationship.

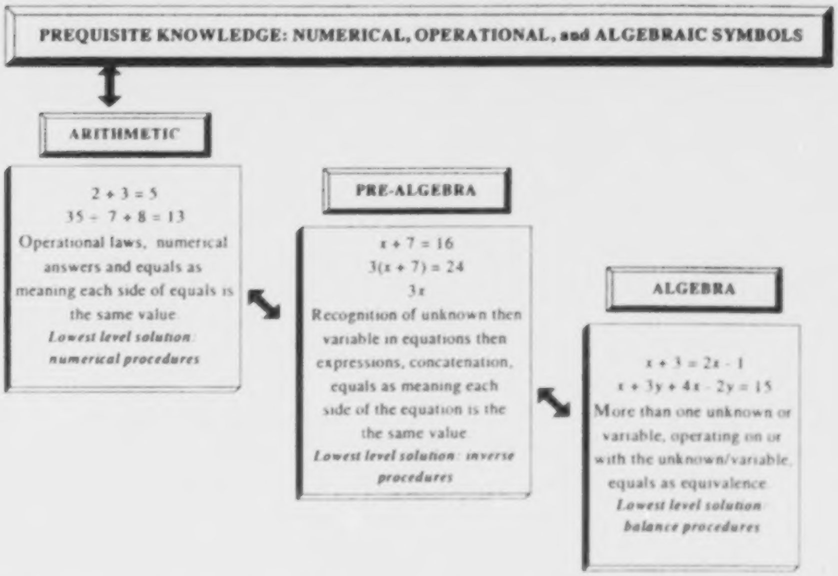

PREQUISITE KNOWLEDGE: NUMERICAL, OPERATIONAL, and ALGEBRAIC SYMBOLS
ARITHMETIC
2 + 3 = 5
35 ÷ 7 + 8 = 13
Operational laws, numerical answers and equals as meaning each side of equals is the same value
*Lowest level solution: numerical procedures*
PRE-ALGEBRA
x + 7 = 16
3(x + 7) = 24
3x
Recognition of unknown then variable in equations then expressions, concatenation, equals as meaning each side of the equation is the the same value
*Lowest level solution: inverse procedures*
ALGEBRA
x + 3 = 2x - 1
x + 3y + 4x - 2y = 15
More than one unknown or variable, operating on or with the unknown/variable, equals as equivalence
*Lowest level solution: balance procedures*

Fig 1. A model of sequential development of algebra knowledge.

The basic assumptions supporting the model are that the developmental literature (Biggs & Collis, 1982; Halford & Boulton-Lewis, 1992; Sfard & Linchevski, 1994) suggests acquisition of pre-algebraic and algebraic concepts in the following order: one occurrence of the unknown in binary operations, a series of operations on and with numbers and the unknown, multiples of the unknown, acceptance of lack of closure and solution with a series of operations on the unknown, and finally relationships between two variables and operations on them.

**Pre-Algebraic Understanding**

Some research in algebra teaching and learning has focused on the transition from arithmetic to algebra and the difficulties in developing algebraic concepts caused by a cognitive gap (Herscovics & Linchevski, 1994) or didactic cut (Filloy & Rojano, 1989). The cognitive gap/didactic cut is located between the knowledge required to solve arithmetic equations, by inverting or undoing, and the knowledge required to solve algebraic equations by operating on or with the unknown. Linchevski and Herscovics (1996) found that seventh graders solved first degree equations such as $ax + b = c$, using inverse operations in the reverse order. For examples involving two occurrences of the unknown there was a shift in procedures as most students used systematic approximation based on numerical substitution. They concluded that students could not operate spontaneously on or with the unknown and that grouping algebraic terms is not a simple problem.

BEST COPY AVAILABLE

They also argued that students viewed algebraic expressions intuitively as computational processes (cf. Sfard & Linchevski, 1994) and in teaching, instead of moving from variable to expression to equation, arithmetical solution of linear equations might be more suitable initially for learning to operate on or with the unknown. Filloy and Rojano (1989) believe such concerns point to the need for an operational level of 'pre-algebraic knowledge' between arithmetic and algebra.

### Cognitive Difficulties in the Transition from Arithmetic to Algebra

It is important for students to understand structural notions in arithmetic, such as commutative and associative laws, as a basis for learning algebra. However MacGregor (1996) suggested that students are unsure of the commutative law while Booth (1988) also reported that students hold an inadequate conception of commutativity believing that division, like addition, is commutative. Demana and Leitzel (1988) maintain that a sound understanding of the distributive property is essential for algebraic functioning. Students also require an understanding of equals as denoting an equivalent relationship. Linchevski (1995) argued that from a psychological point of view, operating algebraically requires students to move from a unidirectional mode of reading an equation to multi-directional processing of information. However students' conceptions of equals has been documented (Booth, 1988; Herscovics & Linchevski, 1994; Kieran, 1981) as indicating an operation to be performed on the left of the equal sign with the answer appearing to the right. Inadequacies in students' arithmetic knowledge base were revealed in a study by Linchevski and Herscovics (1994) involving grade six students. They suggested that appropriate preparation in arithmetic in upper primary school may help to overcome cognitive obstacles and facilitate development of new algebraic skills. Moving from arithmetic to algebra requires students to operate on variables (Filloy & Rojano, 1989) however conceptual obstacles in interpreting letters have been noted. These include a lack of understanding of concatenation (Herscovics & Linchevski, 1994) and students interpreting letters in six different ways including an unknown, a generalised number and a variable (Küchemann, 1978).

This study set out to determine (a) the fit of the model (Figure 1) that depicts the sequential development of algebraic knowledge from prerequisite arithmetic knowledge, to pre-algebraic knowledge, to algebraic understanding and (b) to determine the prerequisite mathematical knowledge that students beginning algebra possessed in order to identity any inadequacies in this knowledge base that may add to the complexity of learning algebra. Results from the first year of the study are presented in Boulton-Lewis et. al. (1997). This paper presents results for the three-years of the longitudinal study.

## Method

### Sample

Initially, the sample comprised 51 grade 7 students from four state primary schools in Brisbane. These were feeder schools for the high school where, in the second year of the study 40 grade 8 students from the initial sample participated and in the third year 33 remaining grade 9 students participated. Analyses were carried out for the 33 students who remained throughout the three years. Three primary schools and the high school were in a middle socio-economic area; the other primary school was in a lower middle socio-economic area.

### Tasks and Procedure

Qualitative methodology formed the basis of data collection and analysis. Interviews were conducted with grade 7 students before any formal algebra instruction took place and grade 8 students after they had received instruction in operational laws, use of brackets, and solution of arithmetic word and number problems. Grade 9 students were interviewed after they had learnt about an 'unknown' in a linear equation and solution of linear equations using backtracking and balance procedures. Students were presented with expressions and equations and asked questions that investigated: for example operational laws; commutative (x, -, ÷, +; which sign/s fit 35 ? 76 = 76 ? 35) and distributive (explain this statement, 6x13 = 60+18); inverse operations (5x71 = 355, 355 ? 5 = 71; 64-29 = 35, 35 ? 29 = 64); order of operations ( $32+\frac{12\text{x}8}{3}$ ), and meaning of equals in an equation (What does equals mean here? 28÷7+20 = 60-36); solution of an equation with one variable ($3x+7 = 22$) and more than one variable ($x+3y + 4x-2y = 0$). Students were interviewed individually and sessions were videotaped.

### Analysis

Interviews were transcribed and analysed to identify crucial knowledge. The computer program, NUD*IST (Richards & Richards, 1994), was used to classify responses. For laws, inverse operations, and order of operations, responses were identified as *satisfactory* or *unsatisfactory* as a basis for learning algebra. Responses for the other tasks were categorised in accordance with the model as depicted in Figure 1 and explained below.

- *inappropriate*, lack of knowledge required for the task; equals as the answer;
- *arithmetic*, recognition and use of arithmetical operations, laws, numerical answers (Sfard & Linchevski, 1994); equals as each side of the equals sign is the same value;
- *pre-algebraic*, recognition and use of unknown, variable, concatenation, and use of inverse procedures to find an unknown in an equation (Herscovics & Linchevski, 1994); equals as each side of the equation is the same value; and
- *algebraic*, recognition and use of relationships expressed in simplified form, (Booth, 1988) employing algebraic processes, such as a balance approach, to solve an equation by operating on or with the unknown; equals as equivalence.

## Results

### Operational Laws, Inverse Operations, and Order of Operations

Table 1 summarises the frequency of satisfactory (S) and unsatisfactory (U) responses. In grades 7 and 8 the majority of students (19 and 17 respectively) could not explain *commutativity* satisfactorily. Many students exhibited a focus on the equals sign and stated that 76 was the answer. However by grade 9, 25 students gave a satisfactory explanation for commutativity. The majority of students in grades 7 (21) and 8 (17) could not give a satisfactory explanation for the *distributive* law and while a substantial number of students (14) still could not explain this law in grade 9, 19 students were able to provide a satisfactory explanation. *Inverse operations* were explained satisfactorily by the majority of students in each grade (26, 30, and 33 respectively). By grades 8 and 9 most students explained *order of operations* satisfactorily (26 and 23 respectively) which was an improvement on only nine satisfactory explanations in grade 7. Unsatisfactory explanations often constituted an incorrect order such as add 32 and 12, times that by 8, then divide by 3.

| Grade | Commutative Law | | Distributive Law | | Inverse Operations | | Order of Operations | |
|---|---|---|---|---|---|---|---|---|
| | x - ÷ +<br>35 ? 76 = 76 ? 35 | | 6x13= ? | | 5x71 = 355,<br>355 ? 5 = 71. | | $32 + \frac{12 \times 8}{3}$ | |
| | S | U | S | U | S | U | S | U |
| 7 | 14 | 19 | 12 | 21 | 26 | 7 | 9 | 24 |
| 8 | 16 | 17 | 16 | 17 | 30 | 3 | 26 | 7 |
| 9 | 25 | 8 | 19 | 14 | 33 | 0 | 23 | 10 |

Table 1. Satisfactory (S) and unsatisfactory (U) responses for arithmetic laws and principles.

### Meaning of Equals

Table 2 summarises the frequency of categories of responses for meaning of equals in the equation $28 \div 7 + 20 = 60 - 36$. The majority of responses moved from arithmetic in grade 7 when 19 students stated equals meant the answer, to arithmetic (12) or algebraic (12) in grade 8 as students explained equals as either the answer or denoting equivalence, to algebraic in grade 9 with most students (19) explaining equals as equivalence or showing a balanced equation.

| | Categories of Responses for Equals in $28 \div 7 + 20 = 60 - 36$ | | | |
|---|---|---|---|---|
| Grade | Inappropriate | Arithmetic | Pre-algebraic | Algebraic |
| 7 | 2 | 19 | 9 | 3 |
| 8 | 1 | 12 | 8 | 12 |
| 9 | - | 5 | 9 | 19 |

Table 2. Frequency of categories of responses for meaning of equals by year.

### Solution of Equations with One Variable and More than One Variable

Table 3 summarises the frequency of categories of responses for solution of an equation with one variable. The majority of students in grades 7 (14) and 8 (13) did not know how to solve $3x+7 = 22$. Nine students in grade 7 and 10 students in grade 8 used inverse arithmetic processes to find the space after $x$ as they believed that $x$ was a times sign. Ten students in grade 7 and 10 students in grade 8 used inverse processes to solve for $x$ which was categorised as pre-algebraic. By grade 9 most students (23) solved $3x+7 = 22$ pre-algebraically by using inverse processes. Two students did not know how to solve the equation, two used an incomplete balance method (pre-algebraic) by taking 7 from both sides, however at this point the students then divided 15 by 3 to get 5. Six students used a complete balance procedure which was categorised as algebraic. The equation $x+3y+4x-2y = 0$, presented to students in grade 9, required simplification then substitution of values for $x$ and $y$. Five students correctly completed this process and 15 simplified the equation (pre-algebraic) to $5x+y = 0$ but could go no further. Four students used trial and error (arithmetic) and nine students did not know how to solve this equation (inappropriate).

| | Categories of Responses | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Grade | Inappropriate | | Arithmetic | | Pre-algebraic | | Algebraic | |
| | $3x+7=22$ | $x+3y+4x-2y=0$ | $3x+7=22$ | $x+3y+4x-2y=0$ | $3x+7=22$ | $x+3y+4x-2y=0$ | $3x+7=22$ | $x+3y+4x-2y=0$ |
| 7 | 14 | - | 9 | - | 10 | - | - | - |
| 8 | 13 | - | 10 | - | 10 | - | - | - |
| 9 | 2 | 9 | - | 4 | 25 | 15 | 6 | 5 |

Table 3. Frequency of categories of responses for equations with one or more variables.

## Discussion

Results of this study revealed that most students in grade 7 did not have a satisfactory understanding, of either commutative or distributive laws, or order of operations that is necessary to operate algebraically. In grade 8 many still could not explain commutative or distributive laws satisfactorily. These results correspond with the findings of MacGregor (1996) and Demana and Leitzel (1988) in that students have inadequate conceptions of these principles. Some students gave inappropriate responses for the commutative law as they focused on the equals sign indicating the answer and stated that none of the signs would fit. Thus students not only failed to see the full structure of the equation, they also failed to see the relationship between elements of the problem. Scandura (1971) argued that algebra is based on relationships, thus these students would not be able to operate algebraically with such limited understanding of arithmetic rules and procedures, particularly of the equals sign. Further to this it becomes apparent that there is a need for students to understand basic numerical properties as some students failed to recognise the distributive relationship between 78 and '6 times 10 plus 6 times 3'. It was not until grade 9 that most students displayed sufficient understanding of commutative and distributive laws that would enable them to apply these to linear equations. Such inadequacies point to the need for explicit instruction in these arithmetic principles in primary school if cognitive difficulties for students beginning algebra are to be reduced.

Most students' perceptions of equals moved from arithmetic interpretations in grade 7 to arithmetic or algebraic in grade 8 to algebraic in grade 9 where most students explained '=' as denoting an equal or balanced relationship. Kieran (1981) noted that students require an equivalence understanding of equals to operate algebraically. By grade 9, 19 students did indicate an equivalence understanding, however there were still 14 students who were operating at either an arithmetic or pre-algebraic level. Thus while students knowledge of '=' had progressed over the years, there was still a substantial number of students who did not understand '=' in an algebraic sense and would need to learn the concept of equivalence. Providing instruction of equals during early arithmetic as indicating each side as being the same value and moving to pre-algebraic instruction of equals where the focus is the 'equation' and equality of sides of the equation, may bridge the gap between arithmetic and algebraic understanding of equals.

Results for solving the equations showed students lacked understanding of concatenated *x*. Herscovics and Linchevski (1994) and Filloy and Rojano (1989) noted that students have difficulty in operating on or with letters in equations and defined this as a gap between arithmetic and algebra. However the results of this study also indicated that, from grade 7, students had a good understanding of inverse operations. In attempting to solve the equations, students often disregarded what they did not understand and subsequently applied a sequential inverse process thereby basing their solution procedures, albeit incorrectly, on what they knew from arithmetic. We propose that sequential inverse procedures constitute a sound basis for learning the procedures of operating on equations with unknowns and would be appropriately placed at a pre-algebraic level of functioning. By grade 9 most students could simplify the equation with two variables and in fact five students understood the variable relationship between *x* and *y* in the equation without being taught.

## Conclusion

Results of this study concur with findings from other studies (Herscovics & Linchevski, 1994; Boulton-Lewis et al., 1997) in that learning to operate algebraically is a complex process that is dependent upon a sound understanding of arithmetic and pre-algebraic principles. Thus the more complex concepts of algebra can develop sequentially. We have conceptualised this as a sequential development of knowledge as depicted in the model in Figure 1. In particular, this study highlights the need for an operational level of pre-algebra, as discussed earlier, to address inadequacies in students' prerequisite knowledge and to prepare students for the symbolism and operations of algebra. Future research should investigate teaching of primary school mathematics to determine cognitive obstacles to understanding operational laws and meaning of equals.

## References

Biggs, J.B., & Collis, K.F. (1982). *Evaluating the quality of learning: The SOLO taxonomy (Structure of Observed Learning Outcomes).* New York: Academic Press.

Booth, L.R. (1988). Children's difficulties in beginning algebra. In A.F. Coxford & A.P. Shulte (Eds.), *The ideas of algebra, K-12.* (1988 Yearbook: 8-19). Reston, VA: National Council of Teachers of Mathematics.

Boulton-Lewis, G.M., Cooper, T.J., Atweh, B., Pillay, H., Wilss, L., & Mutch, S. (1997). The transition from arithmetic to algebra: A cognitive perspective. In *Proceedings of the 21st Annual Conference of the International Group for the Psychology of Mathematics Education, Lahti, Finland,* 2, 185-192.

Collis, K. (1975). The development of formal reasoning. Report of a Social Science Research Council sponsored project (HR 2434/1) carried out at the University of Nottingham, University of Newcastle: NSW. Australia.

Demana, F., & Leitzel, J. (1988). Establishing fundamental concepts through numerical problem solving. *The ideas of algebra, K-12*. (1988 Yearbook: 61-69). Reston, VA: National Council of Teachers of Mathematics.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19-25.

Halford, G.S., & Boulton-Lewis, G.M. (1992). Value and limitations of analogs in teaching mathematics. In A. Demetriou, A. Efklides, & M. Shayer (Eds.), *Neo-Piagetian theories of cognitive development* (pp. 183-209). London: Routledge.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27, 59-78.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School, 7*(4), 23-26.

Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behaviour, 14*, 113-120.

Linchevski, L., & Herscovics, N. (1994). Cognitive obstacles in pre-algebra. In *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*. Lisbon, Portugal, 3, 176-183.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics*, 30(1), 39-65.

MacGregor, M. (1996). Curricular aspects of arithmetic and algebra. In J. Gimenez, R.C. Lins, & B. Gomez (Eds), *Arithmetics and algebra education: Searching for the future*. 50-54.

Richards, T., & Richards, L. (1994). *QSR NUD*IST*. Alladin Systems Inc., California.

Scandura, J.M. (1971). *Mathematics: Concrete behavioural foundations*. New York: Harper and Row.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics*, 26, 191-228.

---

[1] This research was funded by a grant from the Australian Research Council (No. A 79531738).

# STUDENTS' KNOWLEDGE OF LENGTH UNITS: DO THEY KNOW MORE THAN RULES ABOUT RULERS?

**Philippa Bragg and Lynne Outhred**

**Macquarie University, Australia**

*This paper presents the results of a set of tasks designed to examine student understanding of units of measure, ruler construction and the measurement process. While the majority of the 120, Grade 1-5 students were able to use informal units and rulers, a large number of these students performed poorly on tasks requiring them to apply a deeper understanding of units and ruler construction. Few students were able to use informal units to construct a ruler and fewer still could show that numerals on a ruler referred to linear units. The results indicate that teachers should not rely on basic 'read' and 'draw' measuring tasks to assess students' understandings of linear measurement.*

## INTRODUCTION

In the years following Piaget and his co-workers' research into children's measurement concepts (Piaget, Inhelder, and Szeminska, 1960), our knowledge of the way in which students construct their understanding of linear measurement has continued to expand (Boulton-Lewis, 1987). Length concepts are particularly important, as length is usually the first measurement process that students learn about in the early years of school and therefore provides the basis for further development. However, Hiebert (1984) suggests that it is easy to underestimate the complexity of the measurement process, particularly in the first years of instruction.

Students may be taught techniques for measuring lengths, rather than the principles that govern the measurement process. Such techniques may not provide the basis for developing ideas of units and scales or a framework for later measurement concepts. If students do not have an understanding of linear units they are unlikely to succeed in more complex areas of mathematics; for example, area and volume in which length units are extended to two and three dimensions respectively. Ward's work (1979) with ten-year-old students on the construction of graphs has indicated the importance of the fundamental properties of scales. Properties include the point of origin and how units of measurement are iterated.

Evidence from assessments carried out in Britain and America has shown that many students do not have a thorough knowledge of length measurement (Carpenter, Lindquist, Brown, Kouba, Silver, and Swafford, 1988; Hart, 1989). More recently, results of the international comparisons of mathematics (TIMMS) have indicated that secondary students continue to have problems with linear measurement (Lokan, Ford, and Greenwood, 1996). In these papers the kinds of errors that students make on paper-and-pencil assessment tasks have been documented. The source of these regularly reported errors seems to lie in inadequate understandings of the property being measured (Schwartz, 1997, Wilson and Rowland, 1992) and of how to align a ruler (Nunes, Light, and Mason, 1993, Carpenter et al, 1988, Hart, 1982). Problems also occur when students do not understand what is being counted when informal units are used (Wilson and Rowland, 1992).

It has been suggested that introducing measurement instruction with informal units promotes understanding of measuring processes by helping students to see the relationship between the 'continuous' nature of length and the 'parts' that may be counted (Wilson and Osborne, 1988). Teachers should clarify to their students the nature of these parts (or units) that are counted (Hiebert, 1984). Measurement with informal units is formalised into an abstract representation with the introduction of standard unit scales, in particular, rulers. Rulers are "...an indirect method of laying down units of length end-to-end. The numerals on the ruler indicate the number of unit lengths used if they were placed in this manner" (Thompson and Van de Walle, 1985, p.8). However, students may not understand even such a common tool as a ruler.

The distinction between counting discrete objects and the 'counting' of a continuous property defined as length in each measuring task has been emphasised by Wilson and Rowland (1992). This distinction is crucial to understanding the importance of 'zero' in its role as an indicator of no movement in the direction away from the place chosen to start the measuring. The importance of 'zero' in the construction of rulers, tape measures, and student-constructed tools should also be made clear to students (Thompson and Van de Walle, 1985). Familiarity with the process of 'renaming zero' is an important skill when a line cannot be not aligned with zero. A solution can then be obtained by counting units or by subtraction.

As early as 1973, Collis suggested that a student's mastery of a topic was not necessarily indicated by the student's success with basic mathematical techniques (Ainley, 1991). The importance of investigating students' constructions of mathematical topics is supported by the growth of constructivist theory in recent years (Cobb, Yackel and Wood, 1992; Steffe, von Glaserfeld, Richards, and Cobb, 1983). One aspect of the classroom microculture that Cobb (1998) emphasised is the way in which the tools can shape students' mathematical development. The effects of tool use on understanding have also been discussed by Hiebert et al (1997).

The aim of this paper is to investigate the growth of students' knowledge of length measurement using a common measuring tool, a ruler, across the primary school years. The paper reports the results of a subset of ruler and scale questions from a larger study of the development of children's understanding of linear measurement.

## METHODOLOGY

The study was cross-sectional; 120 students from grades 1-5 were selected from four state primary schools in a medium to low socio-economic area of Sydney. Each class teacher selected six students: one girl and one boy considered 'above average', 'average', and 'below average' in terms of mathematical concepts. The first researcher interviewed individual students towards the end of the school year (September-November). Thus, they had been exposed to a large part of the measurement program for their grades. The interview tasks were designed to elicit information about the students' understanding of length measurement. Practical tasks were used because the effectiveness of paper-and-pencil items for the assessment of mathematics knowledge has been questioned (Clements and Ellerton, 1995). These authors' concerns would appear to be particularly pertinent to measurement tasks that are inherently practical.

The subset of questions reported in this study were designed to determine what students knew about rulers. The students were given four common classroom tasks (Tasks 1-4) involving length measurement (the technique of using a ruler), as well as three tasks (Tasks 5-8) that forced the students to apply their knowledge of rulers in unfamiliar contexts (the understanding of using a ruler). Two additional tasks (Tasks 9-10) were given; these assessed the use of informal units—in this case, paper clips—as a measuring tool.

**Table 1 The tasks involving ruler use and knowledge**

| Task | Description | Knowledge |
|---|---|---|
| 1 | Draw a line 7 cm long above a ruler printed on the page (0 is aligned with the end of the ruler). | For an object aligned with a scale, the endpoint is a measure of length only if the other end is aligned with 0 |
| 2 | Draw a 9 cm line with a wooden ruler (0 is not aligned with the end of the ruler). | As for Task 1 |
| 3 | Measure a line (6 cm) shorter than the ruler. (0 is not aligned with the end of the clear plastic ruler). | As for Task 1 |
| 4 | Measure a line longer than the ruler; Grades 1-2 measured 16cm, Grades 3-5 26cm, with the same ruler as for Task 3. | As for Task 1. Length is additive. |
| 5 | Measure a 9cm line with a ruler cut at the 3.5cm mark. Scale marked from 4 to 20cm. | Length can be measured by counting spaces on a unit scale. A numeric scale can be applied to a congruent set of marks. |
| 6 | Measure an 11cm line using a ruler showing centimetre marks without a numeric scale. | As for Task 5 |
| 7 | Draw a 6cm line above a printed "ruler". The marked scale begins at 6. A 3.5cm line extends to the left of the scale. | As for Task 5 |
| 8 | Count five sea horses and state what the '5' represents. Then explain what '5' on the ruler represents and identify a single unit. | Linear units are separated by marks. A numeric scale aligned with the marks gives the number of linear units from the origin. |

108

| Task | Description | Knowledge |
|---|---|---|
| 9 | Use 2 paper clips to measure a line 28cm long, noting a fractional unit. | Iteration of units of constant size with no gaps or overlaps, awareness of a fractional unit. |
| 10 | Make a ruler using paper clips as the unit of measure (students were given a long rectangular piece of light cardboard). | Construction of a scale |

For all tasks except Task 2, clear plastic rulers marked in centimetres were used; these were made from overhead transparencies. For Task 2 a wooden ruler was used; this ruler was different from the one supplied to schools where the zero mark is aligned with the end of the ruler.

The tasks have been grouped to emphasise their conceptual similarities. They were presented in the same order to all students but not presented in order given above.

## RESULTS AND DISCUSSION

The results for the tasks involving the technique of using a ruler are presented in Table 2 for each grade level. There were 24 students at each grade level.

**Table 2 Percentage of correct responses for each grade—ruler technique (Tasks 1 to 4)**

| Task | Grade 1 | Grade 2 | Grade 3 | Grade 4 | Grade 5 | Total |
|---|---|---|---|---|---|---|
| 1 | 37 | 50 | 75 | 79 | 91 | 65 |
| 2 | 17 | 27 | 47 | 79 | 87 | 48 |
| 3 | 8 | 20 | 62 | 72 | 87 | 44 |
| 4 | 0 | 0 | 25 | 67 | 83 | 35 |

The results in Table 2 indicate that by Grade 5 more than 80% of students could measure and draw lines with a ruler. Although ruler use is not usually taught until Grade 2, almost 40% of students in Grade 1 could draw a 7cm line using a ruler. As might be expected, the greatest increase in correct use of a ruler occurred from Grade 2 to Grade 3. Almost all errors were related to ruler alignment. The percentage of the sample making each error is shown in Table 3.

**Table 3 Percentage of origin errors for ruler technique tasks (Tasks 1 to 3)**

| Task | Zero on scale | One on scale | End of ruler |
|---|---|---|---|
| 1 | 65 | 35 | Not applicable |
| 2 | 51 | 29 | 20 |
| 3 | 50 | 38 | 12 |

Younger students and lower performers in Grades 2-3 almost exclusively aligned the ruler with the end, while older students' errors were caused by alignment with 'one'. About a fifth (22%) of the students expressed their confusion about alignment. Although younger students had not had formal training with rulers, it was observed to be the preferred tool when a selection of formal and informal tools was available. Most students (89%) were not able to give a mathematically based reason for the way they aligned; they gave reasons such as: "I always do it like that, Miss.", "Mrs. S. said to start at nought." and, "'Cause it's like counting, see? 1, 2, 3, and it ends on 7." Correct responses however, referred to zero or nought as the 'start' or that the line 'hasn't gone anywhere yet.'

By contrast, the results for the second set of tasks did not indicate that most students understood of the construction of a ruler scale. These results are given in Table 4.

**Table 4 Percentage of correct responses for each grade—ruler understanding**

| Task | Grade 1 | Grade 2 | Grade 3 | Grade 4 | Grade 5 | Total |
|---|---|---|---|---|---|---|
| 5 | 0 | 8 | 20 | 54 | 58 | 28 |
| 6 | 0 | 13 | 21 | 46 | 54 | 26 |
| 7 | 0 | 13 | 25 | 33 | 41 | 22 |
| 8 | 0 | 4 | 13 | 21 | 37 | 15 |

In this set students must use higher-level skills and understandings to solve tasks that rarely appear in student texts or worksheets. Students who were successful in the previous tasks made errors similar to those made by younger students on the first set of tasks; e.g. they counted the first unit mark as 'one', whereas previously they had aligned a ruler correctly. Only slightly more than half the Grade 4 and 5 students could measure a length when the line to be measured was not aligned with zero or when the numeric scale was not shown. Task 7 gave a misleading cue; partway along a line, a scale was marked beginning at 6 and students were asked to draw a 6cm line. Only about 40% of Grade 5 successfully completed this task and its successor, Task 8, which assessed if students could explain what the numerals on a ruler represent.

Few students were able to identify what five on a ruler represented. Student responses could be grouped into the following categories: (1) Points or draws a line to five ("That's where the line ends"), but is unable to show a linear unit (may point to one). (2) Points to or draws over the unit mark for five or counts unit marks but cannot resolve the anomaly when six unit marks are counted. (3) Points to or draws over the unit mark for five, can draw a line to five, or counts five spaces but is confused about what one centimetre is—the space or the mark. (4) Five refers to centimetres, draws a 5cm and 1cm line, or draws a line of 5 identifiable centimetre lengths.

Using informal units to measure a length (see Table 5, Task 9) was a relatively easy task, especially for Grades 4 and 5. It was not done as successfully by younger students as using a ruler to measure. Many younger students were quite inaccurate because they could not iterate the units; they usually slid one or both clips along the length.

**Table 5 Percentage of correct responses for each grade—informal units (Tasks 9 and 10)**

| Task | Grade 1 | Grade 2 | Grade 3 | Grade 4 | Grade 5 | Total |
|---|---|---|---|---|---|---|
| 9 | 11 | 18 | 47 | 83 | 100 | 60 |
| 10 | 0 | 21 | 29 | 54 | 66 | 34 |

Task 10 provided a rich source of information about students' thinking concerning units of measure and ruler construction. Although all students could say that the main features of a ruler are numbers and marks, the detail and precision of ruler construction varied widely. Many students used the paper clip as a unit marker for arbitrary units rather than as the unit of measure. The following construction strategies were observed:

*Paper clips not used:* Young students did not use paper clips in the construction, preferring instead to write a series of numbers with no consistent unit spacing. In Grade 2, students used a series of numbers with some evidence of spacing and the occasional inclusion of unit marks. The constructions of more able students in Grade 2 showed numbers aligned with the unit marks.

*Paper clips used as unit marks:* A large number of students (41%) used paper clips as marks to denote arbitrary-sized units with numerals aligned with the paper clips. Simpler scales had numerals aligned with unit marks and some included zero. More detailed attempts had included zero and some fraction marks for arbitrary-sized units.

*Paper clips used as a unit of measure:* At this level detailed attention to unit size numeral placement and fractions was evident. Some students initially set up the clips in a row but abandoned this arrangement in favour of iteration of a single paper clip. When questioned, all said it was easier to put the unit marks in the right place.

When the three sets of results (Tables 2, 4 and 5) are compared, there appears to be far greater change from Grade 4 to Grade 5 on the ruler-technique and informal-units tasks. However, for the four tasks involving understanding a ruler there is a substantial change (an average of about 20%) from Grade 3 to Grade 4 that flattens out from Grade 4 to Grade 5 (an average of about 10%). More improvement would be expected because only about half the Grade 5 students gave correct responses on these tasks. These results are consistent with teachers focussing on techniques for using a ruler rather than understanding how a ruler is constructed.

## CONCLUSION

These results from this study show that in spite of the ease with which most students measured with rulers and counted informal units, many students did not understand the relationships between linear units and a formal scale. The main errors made in the tasks involving ruler techniques concerned alignment. Such errors might arise from the emphasis on one in counting or, as found by Gravemeijer, McClain and Stephan (1998), confusion arising from methods of counting informal units such as paces.

Students used two main measurement strategies in this study: counting informal units, unit marks, or unit spaces; or aligning the ruler and reading the scale. Correct use of any of these strategies did not indicate that students understood linear measurement in the more complex tasks. An explanation for students' reliance on procedures might be the way that measurement is currently taught. Worksheets and textbook exercises often involve counting informal units or the techniques of ruler use. Such an emphasis on techniques does not develop more abstract concepts, such as knowledge of how a scale is constructed. To develop such knowledge requires practical experiences of measuring and marking units, followed by discussion about key aspects of such student constructed scales. If students do not understand how scales are constructed, they will not have the basic knowledge to relate measurement of length and number lines, nor have the foundation to develop area, volume and other higher order mathematical applications.

While the majority of older students could use informal units to 'measure' a length, the process of counting them appears to have resulted in different interpretations from those expected by teachers. The emphasis on counting may obscure the *linear nature of the unit of measure* if it is not made explicit when informal units are introduced. A possible explanation would be that students attend to the action of placing or counting discrete units rather than their lengths. According to Wilson and Osborne (1988) neither zero nor the iteration of line segments can be made explicit when informal units themselves are counted, thus reducing the possibility that students are able to make the important link with the underlying linear unit concept (Hiebert, 1989).

In 1984 Hiebert pointed out that "...many elementary school students are quite proficient in a variety of standard measuring skills but lack an understanding of some of the basic concepts involved." (p.19). In 1990 Webb and Briars wrote that teachers should know firstly that students are able to "...reliably and efficiently... " (p.11) apply procedures, but more importantly they should know "...what a student knows about the concepts that underlie a procedure..." (p.11). This research has found that little has changed; few of the students interviewed could reliably demonstrate a deeper understanding of the concepts that underpin linear measurement. These findings have significant implications for teaching about the relationship between informal units and the construction and representation of linear units in formal measurement.

## REFERENCES

Ainley, J. (1991) "Is there any mathematics in measurement?" In The Open University (Ed.), *Teaching and Learning School Mathematics*, (pp.69-76) Milton Keynes: Hodder and Stoughton.

Boulton-Lewis, G. and Halford, G., (1990). An analysis of the value and limitations of mathematical representation used by teachers and young children. In G. Booker, P. Cobb and T. de Mendicuti (Eds.), *Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Education*. Mexico: CONACYT.

Carpenter, T., Lindquist, M., Brown, C., Kouba, V., Silver, E., and Swafford, J. (1988). Results of the fourth NAEP Assessment of mathematics: Trends and conclusions. *Arithmetic Teacher*. December, 38-41.

Carpenter, T. (1976). Analysis and synthesis of existing research on measurement. In R. Lesh (Ed.), *Number and Measurement* (pp47-83). Athens Georgia: ERIC/SMEAC, University of Georgia.

Clements, M. A. and Ellerton, N. (1995). Assessing the effectiveness of paper-and-pencil tests for school mathematics. In W. Atweh and S. Flavel (Eds.) Galtha. *Proceedings of the Eighteenth Conference of the Mathematics Education Research Group of Australia*. Darwin: Program Committee.

Gravemeijer, K., McClain, K. and Stephan, M. (1998).Supporting students' construction of increasingly sophisticated ways of reasoning through problem solving. In A. Olivier and K. Newstead (Eds.), *Proceedings of the Twenty Second International Conference for the Psychology of Mathematics Education*. Stellenbosch: Program Committee.

Hart, K. (1982). *Children's Understanding of Mathematics*: 11-16. London: John Murray.

Hart, K. (1989). There is little connection. In P. Ernest (Ed.). *Mathematics Teaching: The State of the Art*. (pp 138-142). Barcombe, UK: Falmer Press.

Hiebert, J, Carpenter, T., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., and Human, P. (1997). *Making Sense: Teaching and Learning Mathematics with Understanding*. Portsmouth, NH: Heineman.

Hiebert, J. (1989). The struggle to link written symbols with understandings: An update. *Arithmetic Teacher*, 36 (8), 38-43.

Hiebert, J. (1984). Why do some children have trouble learning measurement concepts? *Arithmetic Teacher*, 31 (7), 19-24.

Lokan, J., Ford, P., and Greenwood, L. (1996) *Maths and Science* On the Line. *Australian Junior Secondary Students' Performance in the Third International Mathematics and Science Study*. The Australian Council for Educational Research Ltd. Melbourne, Victoria.

Nunes, T., Light, P. and Mason, J. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction*, 3, 39-54.

Orton. A., and Frobisher, L. (1996). *Insights into Teaching Mathematics*, Cassell, London.

Schwartz, S. (1995). Developing Power in Linear Measurement. *Teaching Children Mathematics, 1*, (7), 412-416.

Steffe, L. von Glasersfeld, E., Richards, J., and Cobb, P. (1983). *Children's Counting Types: Philosophy, theory, and application*. New York: Praeger Scientific.

Van de Walle, J., and Thompson, C. (1985). Let's do It: Estimate how much. *Arithmetic Teacher, 32* (8), 8-12.

Ward, M. (1979). *Mathematics and the 10-year-old*. London: Evans.

Webb, N. and Briars, D. (1990) Assessment in mathematics classrooms, K-8. In T. Cooney (Ed.). *Teaching and Learning in the 1990's*. NCTM Yearbook.

Wilson, P. and Osborne, A. (1988). Foundational ideas in teaching about measure. In T. Post, (Ed.), *Teaching Mathematics in Grade K-8. Research Based Methods*, (pp. 78-110). Boston: Allyn and Bacon.

Wilson, P. and Rowland, R. (1992). Teaching Measurement. In P. Jenner, (Ed.), *Research Ideas for the Classroom. Early Childhood Mathematics*, (pp. 171- 191). New York: Macmillan.

# BECOMING MORE AWARE: PSYCHOANALYTIC INSIGHTS CONCERNING FEAR AND RELATIONSHIP IN THE MATHEMATICS CLASSROOM.

Chris Breen, School of Education, University of Cape Town, South Africa

## Abstract

*The data for paper is drawn from a current research project which aims at exploring the suitability of the Discipline of Noticing (Mason 1994) as a research methodology for assisting teachers improve their teaching practice. In the paper the author describes two particular incidents from his own mathematical classrooms and then explores possible insights that the literature on psychoanalytic practice is able to give to these examples.*

## A. Introduction.

> There are lots of useful observations in the educational literature about learners learning but not so many about teachers teaching. This must be partly because it is so difficult to give an honest account of what it is actually like to teach – most attempts to do this slide into idealised intention or pious hope. (Tahta 1995)

This paper arises out of the pilot stage of a funded research project[1] which aims to explore the possibilities of using the Discipline of Noticing as developed by Mason (1994) as a research methodology for practitioners wishing to improve their own teaching. The Discipline of Noticing requires the practitioner to give accounts-of incidents which occur in the classroom and then to make these accounts available to others so that multiple stories can be based on the account, each of which allows the practitioner to consider alternative readings of the moment as well as different possibilities for action in the future. This report focuses on two such incidents drawn from the author's own teaching of two specific mathematics classes during 1999.

The first class consisted of 11 postgraduate students who had registered for the preservice primary school teaching diploma and who were thus compelled to take the author's Method and Content of Mathematics classes. This meant that they attended two 90 minutes sessions each week throughout the first semester period of 12 teaching weeks.

The second class was a group of adults who had signed on voluntarily for a Winter School class run by the Department of Extra Mural Studies called "Second Chance Mathematics" which was advertised as a "course intended to give those whose school experience led them to believe that they could not do mathematics, a second opportunity to engage with the subject. These include adults who have blocks about doing mathematics and whose level of official 'achievement' is less than a higher grade school pass. No previous knowledge of mathematics above standard five will be assumed". Although a limit of 20 was placed on the course a class of 22 attended four two hour weekly sessions.

---

[1] The financial assistance of the National Research Foundation towards this research is hereby acknowledged. Opinions expressed in this paper and conclusions arrived at, are those of the author and are not necessarily to be attributed to the National Research Foundation.

While the courses were obviously aimed at audiences with different needs and backgrounds and the contact time for one was 8 hours in comparison to 36 for the other, the approach to covering the mathematical content of basic algebra and arithmetic was similar and was based on an approach which had evolved over a period of 15 years of running the Primary pre-service module. The first session was spent getting students to talk about their past mathematical experiences by writing their Hopes and Fears for the course on the board. A simulation drama exercise (Breen ) where an authoritarian teacher called Mr Smith took the class for a maths test was used to assist this process of recalling past experiences of mathematics and students in both classes were asked to keep reflective journals[2]. The entry into the mathematical content was made through a Number Pattern investigation using matchsticks as apparatus where students were encouraged to develop a variety of visual methods to solve the problem. Each different approach was valued and attended to and students were allowed to select their own preferred way of visualisation as more complex problems were tackled.

### B. Noticing Fear.

> The purpose of the data is not to convince someone else by logical argument, but to sensitise and attune oneself (at first) and then later, other people, to notice (Mason, 1994, 27).

The first selected moment did not require any subtlety in noticing technique as it made its present felt in a striking fashion.

*I was in the middle of the third session with the primary group and they had seemingly mastered the basics of developing a formula for the nth term of an arithmetic progression of an apparatus-based problem using their own visualisation method. I had to make a strategic decision whether or not to continue the lesson by giving them a similar example which would allow them to re-confirm their growing mastery or to give them something where they could use their new-found knowledge to move forward. I decided on the latter, and said "I think I'd like to give you a challenge now". Their response was immediate. I had earlier in the year introduced them to the concept of their own personal a tension quotient and they had each reported that it had dropped to manageable levels at the start of this session. Immediately called out that the mere fact of my saying it was a challenge had raised their tension quotients off the scale! Although the first part of the new problem was extremely straightforward and was the same as work they had already done (the challenge I had spoken of was to come in the second stage!), the class struggled*

---

[2] The journal forms an integral part of the pre-service course. The course handout states " In order to facilitate the reflective process necessary to establish your views on teaching mathematics, you must keep a diary of thoughts, ideas, understandings and actions which stem from the course during the first semester.

*enormously. Diana, in particular, became noticeably agitated and flustered, talking more loudly and suddenly burst into tears. Although she stayed in the room, she was unable to talk about the incident until the following day when she explained that the word 'challenge' had caused the turmoil and made her feel all her old insecurities.*

This turned out to be only one of at least three occasions where a student burst into tears while doing mathematics in this class. It should be remembered that each student had already obtained an undergraduate degree. While it might be a fruitful task to explore the ways in which the teacher's methodology contributed to the onset of tears, I prefer to focus on the aspect of fear of mathematics for the purposes of this paper.

The fact that there were three examples of breakdown in this mathematics classroom suggested that the problem might be widespread. Extracts from each of the 11 student journals provided a frightening picture of the damage caused by the class's mathematical experiences (even of those who had obtained good passes in mathematics at the school leaving stage). Consideration for the limitations of available space will force me to restrict myself to two extracts from journals, the one from a pre-service primary teacher with an Honours degree band the other from a PhD participant in the Second Chance Mathematics course.

Extract One: *Session Three dawned with new feelings of anxiety, and the additional fear of feeling foolish in front of everyone else who (I assumed) certainly would fare better than I in this lesson. And certainly in a peculiarly common manner the self-fulfilling prophecy manifested in a lot of careless, basic mistakes, made while actually trying to cope with feelings of inadequacy. The maths almost becomes secondary. The responses triggered by merely being in this environment are astounding... What is strikingly evident is the role of personalities within this context. I think the maths teacher has a profound responsibility to his/her pupils. He/she is in fact a major player in the development of self-concept – and look how most of us look! Fearful, nail-chewing, shivering idiots – which carries over into all areas of life until it becomes so internalised that it becomes an integral and accepted part of our own self-image.*

*How amazing to think that maths has such power. In fact, not even maths itself, but the very thought of it. It is true to say that maths brings out every insecurity and vulnerability that we experience in our lives as a whole. We drag everything into maths – our whole life is laid bare and our defencelessness is exposed when we are forced to attempt to grapple with concepts that are beyond us. This learning maths is a painful process. It is not only about not grasping what is perceived as difficult concepts, but it is about myself, my very being, all that I am and all that I can offer. There is not more. When all is said and done, that is it. That is me. So subsequently I is reduced to someone almost unacceptable. I say almost, because it cannot be that this subject holds such power, yet for some, it is sadly true that they become victims – slaves of a foolish, inconsequential form of logic. And their lives are destroyed by it.*

Extract Two: *But what was/is it about maths that so flaws me and floors me, that leaves me feeling so numb, so absent (as if I am aware of not being here), so fearful, so incapable, so incompetent, so childlike, so unable to function like an adult, so unable to remember things (formulae, procedures), so unable to be the person who I am for myself and for others in the rest of my world, so unable to construct internally coherent analyses that help myself and others to understand the complex relationships between (theories of) literature, (theories of) society and (theories of) psychoanalysis, so unable to think? All of these experiences and self-images came back to me when we worked on the second puzzle - with the matchsticks. I could not work out how to work out the problem - my mind just seized up, it went blank, I could not concentrate, something inside me was saying this is meaningless you don't understand this you can't understand this run away. When I could not concentrate I became even more anxious, blindly/irrationally grasping at the formulae, unable to understand them, copying them down incorrectly, becoming trapped in a vortex of ignorance. (I know this seems like a purple passage, but I want to communicate a sense of being caught in a downward spiral).*

These student writings convey an overwhelming sense of fear and anxiety engendered by their encounters with mathematics. They are also both very clear that it is the subject of mathematics that brings them to this position of fear. For me, their language of experience resonates with those of victims of other forms of abuse. In the face of these strong emotions it seems that any attempt to try to teach them mathematics has to start by acknowledging and addressing this past. While the structure of the course in question seems to have allowed this fear to surface, questions remain as to how this should best be done and what strategies should be used once the fear has been expressed. The important first step in attending these questions was to survey the literature to determine what research has been conducted into this problem.

The earlier reference to the psychological support given to victims of abuse suggests that the annual conference proceedings of the Psychology of Mathematics Education (PME) organisation should be a promising starting point for such research. However it quickly becomes apparent that the use of the word Psychology in PME is, in practice, almost exclusively restricted to cognitive-oriented educational theory rather than to any psycho-analytic or -therapeutic theory. The links between psychoanalysis and mathematics education seem to have largely been silent themes at PME with only a few discernible exceptions since 1992 (Pimm 1994, and Baldino and Cabral 1998), and neither of these directly address the encountered dominance of fear in the mathematics classroom. The only PME paper which promised to relate to the strong feelings of the students towards mathematics was one by Vinner (1996) entitled "Some Psychological Aspects of the Professional Lives of Secondary Mathematics Teachers - The humiliation, The frustration, The hope". Here, the author got a group of teachers together to discuss answers that they had given to a short questionnaire about difficult topics that they teach and he asked them how they coped with these difficulties. He later commented on the similarity of these sessions with group therapy but did not take the matter further.

However, fear of mathematics was the dominant feature of an article by Early (1992) in which he described his experiences in teaching a university mathematics course. As an optional additional credit assignment, Early invited his students to think of a recent mathematics problem which had challenged them and then to find and write about fantasy images which might capture their feelings. He was surprised at the strength of the images (which included frightening life-and-death situations) and subsequently analysed the submitted material using a theoretical framework based on the metaphors of alchemy (as used in Jungian psychodynamic theory) to capture the process of taking ignorance and turning it into gold. In the article, he describes how the various student images and stages of grappling with their mathematical difficulties parallel alchemic processes such as the burning off of impurities/emotional conflicts (calcinatio); the ascending process and liberation of the spirit (sublimatio); the reductive loosening of structure (solutio); and the darkness involved in the shattering of rigidity (mortificatio). Examples of different student images are given for each of the processes and the point is made that in Jungian theory there is no set order in which these various stages should take place, but each has to be honoured and respected. Early stresses that his aim is not to produce better mathematics students but rather to assist them in developing a deeper, more imaginative perspective from which to view their mathematical experience and perhaps their lives.

The incident and the extracts from the student journals have attuned me to the degree of fear that exists in these two mathematics classes, and my hope in writing the paper is that others will have noticed similarities from their experience. Nevertheless, it seems clear that, if fear of mathematics is a serious factor in any class, it is an area that needs to be entered into carefully given the limited amount of relevant research that seems to exist. While the courses from which the data has been taken seem to provide sufficient opportunity for the students to express their feelings (given the readiness of students to communicate through their journals), Early's article suggests that the task of handling that fear and moving beyond that to the successful solving of problems is a complex and lengthy process. It might be a useful starting point to identify classroom possibilities that provide opportunities for development that parallel the different alchemic stages described in the article.

### C. The double bind of Relationship?

> Conducting research is like swimming in a large pool. There is no beginning and end, just water and endless shoreline. One gets in, moves around in the water, and after a while gets out. (Appel 1999, 146)

The second incident occurred in the last session of work with the Second Chance Mathematics class and, while it continues the theme of fear in that the students self-select themselves as having had bad previous experiences with mathematics, the focus changes from the student attitude to the teaching contract.

*I had given the class some homework to explore the patterns involved in the 11 times table. I asked for feedback as to what people had found out overnight and was pleased to note that some new voices were keen to report. One student, Nothemba gave the first part of an answer and waited for me to ask for more before continuing. I commented on the fact that this was a helpful way to answer in that it gave me the possibility of asking others to report their findings, and compared it to the norm where students usually were so excited with their own findings that they want to blurt it all out and, in so doing, spoil it for the others. One of the students, Tessa had been very excited with her explorations on the 9 times table the previous night and initially had her hand up at the start of this lesson indicating that she wanted to contribute but then withdrew it and sat in her chair with her hands folded looking away from the chalkboard. When I approached her to ask what was happening, she replied, "It's OK, I've got your message loud and clear". On two other occasions during the lesson I tried to engage with her to find out what had gone wrong, but she responded each time that she was fine and continued to remain uninvolved with her arms folded until the end of the session, when she joined the rest of the students in thanking me for the course.*

The immediate question that comes to mind is what did I do wrong? How can I undo the damage? Tessa had changed in front of my eyes from an enthusiastic and excited participant to a rebellious and defeated student. The reflections in the earlier first student extract which commented on the profound responsibility of the teacher come back to haunt. In the absence of further information the teacher is left to construct different narratives which may account for the student's response. Various hypotheses come to mind. For example, the teacher might have been irritated at the way that Tessa had been showing off in the class and had looked meaningfully at her when praising Nothemba's behaviour in order to put her in her place; or perhaps the teacher had led Tessa to feel that she was special in the class and so she was hurt when she was not called upon to show the results of her homework off to the class.

Wilson (1994) and Tahta (1995) provide an excellent example of this way of working in their discussion of an initial incident which had been described by Wilson. In his original analysis, Wilson had been drawn to the concept of transference (the imposition of an actual or imagined previous relationship onto a present one) in trying to understand his actions, but the response from Tahta draws attention to other possible interpretations which might draw on concepts such as counter-transference (picking up and being influenced by what another is transferring), projection (disowning the unapproachable parts of oneself and placing them on another) and projective identification (being influenced and reacting according to another's projection).

The problem here is that these narratives have moved beyond the supposedly observable realities of the classroom interaction and into the realm of the unconscious. This is an area on which Blanchard-Laville (1991) concentrated her research. She believes that in the teaching situation there is a subject (in the psychoanalytic sense of subject of the unconscious) present. When making decisions in the teaching situation, the teacher is subjected to internal pressures acting upon him from the unconscious without his knowledge. This leads to a process whereby the teacher repeats former experiences without recalling the prototype and with the very real impression that on the contrary this is something completely motivated by the present moment. Based on her findings, Blanchard-Laville developed a system for training mathematics teachers inspired by the work of the Balint group, in which they do psychic work linked to their professional practice. The group of teachers meet for 2 hours every 2 weeks and focus on the psychological implications of their own accounts of incidents with a view to gaining access to their unconscious. Her work leads her to believe that it is possible for a teacher to develop beyond the reach of what can be achieved through pedagogical guidance or the teacher's own will to change. Teachers have developed a capacity to identify what is at stake for them in classroom episodes and hence become less "split" inside themselves and more flexible and alive in their exchanges with pupils.

These musings need to refer back to the incident described. There is a block in the student's progress that can either be confronted or ignored. Different interpretations will lead to different possibilities for action. Choosing in the moment to act in a certain way requires two things: noticing a possibility to choose, that is recognising some typical situation about to unfold; having alternatives from which to choose, that is, being aware of alternative action or behaviour. The difficulty is that a great deal of the information is inaccessible. In addition, Early's identification of different stages suggests that the rejection of the teacher and his/her authority (mortificatio) might form a necessary part of the search for coagulatio ( the highest form where the solution is reached) the death of the teacher and his/her authority.

### D. Conclusion

> People don't understand what it (teacher development) is like. I tell people that this is a lot more like psychoanalysis than it is about telling somebody a new recipe. (Robert Davis 1992)

One result of the shared communication between Tahta and Wilson is that Wilson comments how unusual it is in teaching for a teacher to be able to examine such things and Tahta responds by recommending a case study group with or without a leader. Tahta makes the further point that therapists and counsellors are trained to try to sort out what's in them and what's in the client. If psychodynamic notions are to be invoked in classroom accounts then standard reflective procedures common to most therapists and counsellors should also be considered. People who wish to address the emotions which are stirred in classrooms need to have the courage to expose their own feelings, but they will also need to be able to sift through various interpretations of them and produce specific reasons why they come to the conclusions they do.

This brings to mind the chapter by Clarke (1988) in which she contrasts her training as a teaching with that of her preparation as a gestalt therapist. She comments on the many similarities of the tasks and skills expected of each especially in their dealings with those who are not coping. She highlights the lack of a focus on process in the preparation of teachers especially with regard to work on self.

The incidents described in this paper have explored one aspect of the teaching of mathematics and have suggested the need for further work to be done to identify and research the contributions that a knowledge of psychoanalytic processes can make to understanding and improving the teaching of mathematics.

### E. References

Appel, S. The Teacher's Headache. In S. Appel (ed.) (1999) *Psychoanalysis and Pedagogy*. London: Bergin &Garvey.

Baldino, R and T. Cabral (1998) Lacan and the school's credit system. In A. Olivier and K. Newstead (eds.) *Proceedings of the 22nd International Conference for the Psychology of Mathematics Education. Stellenbosch, South Africa.*

Blanchard-Laville, C. (1992). Applications of Psychoanalysis to the Inservice Training of Mathematics Teachers. *For the Learning of Mathematics*, 12, 3, 45-51.

Breen, C. (1991) Concerning Mr Smith and his (very brief?) reign of terror. *Pythagoras*, 25, 31-37.

Clarke, R. (1988) Gestalt Therapy, educational processes and personal development. In D. Pimm (ed.) *Mathematics, Teachers and Children*, Hodder and Stoughton: London

Davis, R. (1992) Teachers: The Key to Success. *ETS FOCUS*, 27, 18-20.

Early, R. (1992) The Alchemy of Mathematical Experience. *For the Learning of Mathematics*, 12, 1, 15-20.

Mason, J (1994) *Researching from the Inside in Mathematics Education - locating an I-you relationship.* Expanded version of a plenary presentation to the 18th International Conference for the Psychology of Mathematics Education, Lisbon, Portugal, Milton Keynes: Centre for Mathematics Education, Open University.

Pimm, D. (1994) Attending to Unconscious Elements. *Proceedings of the 18th International Conference for the Psychology of Mathematics Education*. Lisbon, Portugal.

Tahta, D. (1995) Ever present affectivity. In *Chreods 9*, http:/s13a.math.aca.mmu.ac.uk/Chreods/Issue_9/DickTahta/Dick.htm

Vinner, S. (1996) Some Psychological Aspects of the Professional Lives of Secondary Mathematics Teachers - The humiliation, The frustration, The hope. *Proceedings of the 20th International Conference for the Psychology of Mathematics Education*. Valencia, Spain.

Wilson, D. (1994) The Transference Relationship in Teaching. In *Chreods 8*, http:/s13a.math.aca.mmu.ac.uk/Chreods/Issue_8/Dave_W/Dave_W.htm

# SAME/DIFFERENT:
# A 'NATURAL' WAY OF LEARNING MATHEMATICS

Laurinda Brown, University of Bristol, Graduate School of Education, UK

Alf Coles, Kingsfield School, South Gloucestershire, UK

*We have linked pupils' 'needing to use algebra' with the asking and answering of their own questions related to the structure of the context in which they were working (Brown and Coles, 1999). How do such pupil motivated questions arise? We realised that pupils' questions occurred when they were exploring difference or sameness, which we call same/different. We use an enactivist framework and methodology to discuss a project[1] considering the use of same/different as a planning and teaching perspective for 4 teachers of mathematics in secondary schools in the UK, which seeks to describe the emerging samenesses and differences in their classroom cultures. In this paper we illustrate how the teaching perspective same/different is used and how the pupils, over time, 'naturally' use same/different themselves to generate questions such as 'why does this always work?' which they answer structurally.*

## Background

In 1998/9 we worked as a teacher-researcher pair in Alf's classroom with pupils aged 11-12 years old (year 7), Laurinda visiting weekly, on a Teacher Training Agency (TTA) project in the UK to explore Sutherland's (1991) challenge:

> Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas? (p.46).

During that year we worked on ways to characterise pupils' 'needing to use algebra' and saw this as linked to them being able to ask and answer their own questions related to contexts. We reported in one case study about a pupil using algebra in response to a numerical problem fifteen weeks after the start of the project:

> Alex clearly shows evidence of insight into the structure of the problem ... The difference that strikes us here ... is that the algebra has arisen from a question of Alex's ... As he worked through the general case the structure of the problem was illuminated ... we would argue that Alex's need for algebra came through the posing of his own question: *why?* and that this came out of a pattern spotted ... after the process of doing a few examples (Brown et al, 1999, p.159).

What happens in 'the process of doing a few examples'? Is it possible to say more about this? Following from this case study we became interested in how to support pupils in asking their own questions, within contexts in the mathematics classroom, to create new insights into the structure of the problems they were working on. In reflection after lessons we became aware of a common strand of classification activites which seemed powerful in allowing pupils to ask questions eg when there were two examples to contrast or when pupils had a disagreement about what they saw or when they wrote an algebraic formula in many seemingly different ways. We started to refer to both what the pupils were doing and the activities which supported the behaviour as same/different. In turn this awareness of same/different started to inform our planning. It seemed that investigating sameness led to the question 'Why?' of proof or 'convince me?' and difference led to a further exploration of structure.

## Same/different

In observing the pupils' work in response to comparing and constrasting two or more instances, what was striking was the motivation that they had for the tasks. They were able to work from their own observations. St. Julien (1997) makes the connectionist claim that *pattern recognition is the basic mental process and, as such, is the ground for all that we call learning* (p. 275). He illustrates this claim with an activity of classification of single-celled organisms where, without any formal knowledge of their characteristics it is possible to classify them by eye (without thinking about it too much!). This difference motivates the classification rather than the more traditional rote-learning or memorising of the characteristics and then attempting to classify which has a lower success rate.

No matter what authors' epistemological stances, whether accepting of a reality to be known, through to cognition being embodied action, there seems to be a general agreement that our basic way of engaging with the world is through same/different. Inevitably authors use different terminology; egs Skemp (1986) discusses abstracting and classifying in concept formation (p.21), Vosniadou and Ortony (1989) consider similarity and analogical reasoning and Bateson (1978) discusses difference and the 'news of difference':

> It is only *news of difference* that can get from the territory to the map, and this fact is the basic epistemological statement about the relationship between all reality out there and all perception in here: that the bridge must always be in the fornrof difference (p.240).

If same/different is how we learn in the world then why is it that the model of teaching as transmission, where pupils have to rely on their memories, is standard? What would alternatives be like? We were successful in a bid to the ESRC[1] to extend the TTA project into 3 more classrooms in the UK during the academic year 1999/2000 to look at this latter question. The aims are:

> 1) To create year 7 mathematics classroom cultures which provoke a need for algebra.
> 2) To investigate the similarities and differences developed in each of the teacher's classrooms.
> 3) To investigate the nature and extent of the support needed from the collaborative group of teachers to plan their classroom activities starting from the students' powers of discrimination.
> 4) To develop theories and methodologies to describe the complex process of teaching and learning.

The influence of same/different on the aims for this bid is apparent and we believe that all individuals have powers of discrimination which allow them to learn, to make sense of the world they perceive, through an awareness of difference. What we experience through our actions is an interpretation based on all of our past. Therefore two people cannot see the same thing nor share the same awareness. We can communicate, however, because we can talk about the details of common experiences, exploring differences, and in doing so the gap between interpretations can be reduced. What would pupils do in response to their teachers' planning and teaching using same/different? We were still investigating pupils' 'needing to use algebra' and considered that teaching and research strategies would develop throughout the project.

As researchers we would also be learning through same/different in a process described by Engeström and Cole (1997) as *situated interventionism* which *calls attention to disturbances and discoordinations as indications of new possibilities in practice ... and is not satisfied with observing and analysing situated practices; it is engaged in creating new forms of practice* (p.308).

Learning through the process of the doing of the research places us as enactivist (Reid, 1996, Brown and Coles, 1997, 1999, Hannula, 1998). In what follows we describe how the ESRC project was constructed to give insights into the aims above, then link the four principles of our enactivist theoretical frame to methodology and methods.

## Theoretical frame, methodology and methods

Over one academic year, September, 1999 to July, 2000 which is split into 3 terms we (3 teachers, 1 teacher-researcher and 3 researchers) are investigating the samenesses and differences in the developing algebraic activity in the classroom cultures of the 4 teachers through:

- working in a collaborative group, meeting once every half-term for a full day and corresponding through e-mail

- videotaping each teacher for one lesson in every half-term and researchers observing teachers in the classroom at most once a fortnight in teacher/researcher pairs

- every half term interviewing a) each teacher and b) 6 of each teachers' pupils in pairs, selected to give a range of achievement within the class

- encouraging pupils to write a) in doing mathematics and b) at the end of an activity, about 'what have I learnt?': photocopies of all the 'what have I learnt?'s are collected from each teacher aswell as all the written work of the 6 pupils interviewed

- each researcher being responsible for viewing the data collected through one or more strands; teacher strategies, pupil perspectives, algebraic activity, samenesses and differences in the classroom cultures, teachers use of same/different in planning to teach through pupils using their powers of discrimination.

The classroom cultures were developed through the teachers sharing with their pupils that the year was about them 'becoming mathematicians'. This structure is to support our looking at what pupils and teachers do in these classrooms. Schemes of work and organisational structures within the schools are different and it is not our intention to change these. The content of the lessons would still be decided by the teachers within those structures but during the day meetings there would be time to plan together, given those constraints, to allow pupils to use their powers of discrimination.

*A first principle of enactivism* is the recognition that we cannot take in the details of everything that is happening around us. We are naturally selective since there is a limited capacity to what we can attend to. What we notice and the connections we make guide our actions, often implicitly. It is in this sense that cognition is placed as being 'perceptually guided action' or 'embodied actions' (Varela, 1999, p.12, p.17).

We are therefore giving space within the classrooms for the pupils to work at making connections and for them also to communicate these to the whole class. The teachers make their decisions contingently upon what they perceive through their awareness of samenesses and differences within what the pupils are sharing. The teacher cannot be in control of the content nor hear and respond to everything that is happening in such classroom interaction. We try to set up the possibility of the pupils making connections through common boards used for sharing questions, conjectures, homework etc.

*The second principle of enactivism* (adapted from Varela, 1999, p.10) is the belief that we *are* what we do. It is our actions and perceptions that make us who we are and these are dependent on the whole of our past experiences. Consequently the data collection on the project is done over time and we are working with pupils who have just started at a new school to establish new behaviours in their mathematics lessons related to 'becoming mathematicians'.

*The third principle* we adopt is that we take multiple views of a wide range of data:

> The aim here is not to come to some sort of 'average' interpretation that somehow captures the common essence of disparate situations, but rather to see the sense in a range of occurrences, and the sphere of possibilities involved (Reid, 1996 p.207).

Multiple views of the data are captured through the different strands which the researchers are investigating. We take one incident and interpret it through different strands and also tell stories of the changes that are happening over time. One powerful way of working with multiple views is through the use of the videotapes, using short extracts and talking through the details of what we see. What seems important is that overlapping themes emerge over time.

*As a fourth principle,* from these overlapping and interconnecting themes, theories which are 'good enough for' (Reid, 1996) a purpose emerge:

> theories and models ... are not models of ... they do not purport to be representations of an existing reality. Rather they are theories for; they have a purpose, clarifying our understanding of the learning of mathematics for example, and it is their usefulness in terms of that purpose which determines their value (Reid, 1996 p.208).

There is no sense of there being a 'best' theory for our work. Our theories are 'good-enough for' our actions and the ideas that we continue to think about and use as teachers and researchers are those that inform our practice. We recognise what is useful for the practice of our teaching by what we are doing in the classrooms. What is not useful does not happen. We see our *research about learning as a form of learning* (Reid, 1996 p.208) where our learning is gaining a more and more interconnected set of awarenesses about our teaching of mathematics. These are not the explicit products of this research, however, what is crucial here is the *process* of developing such theories and actions in the classrooms.

We now present evidence for the developing use of same/different within this project which is supporting pupils' developing use of their own questions in working mathematically, 'needing to use algebra'.

**Illustration of common board and teacher use of same/different**

At the start of a lesson the pupils' responses to homework are commonly shared on the boards. In this typical illustration, Teacher A organised the pupils to share shapes that they had drawn for homework with order of rotational symmetry four. At the end of the lesson the board looked like this:

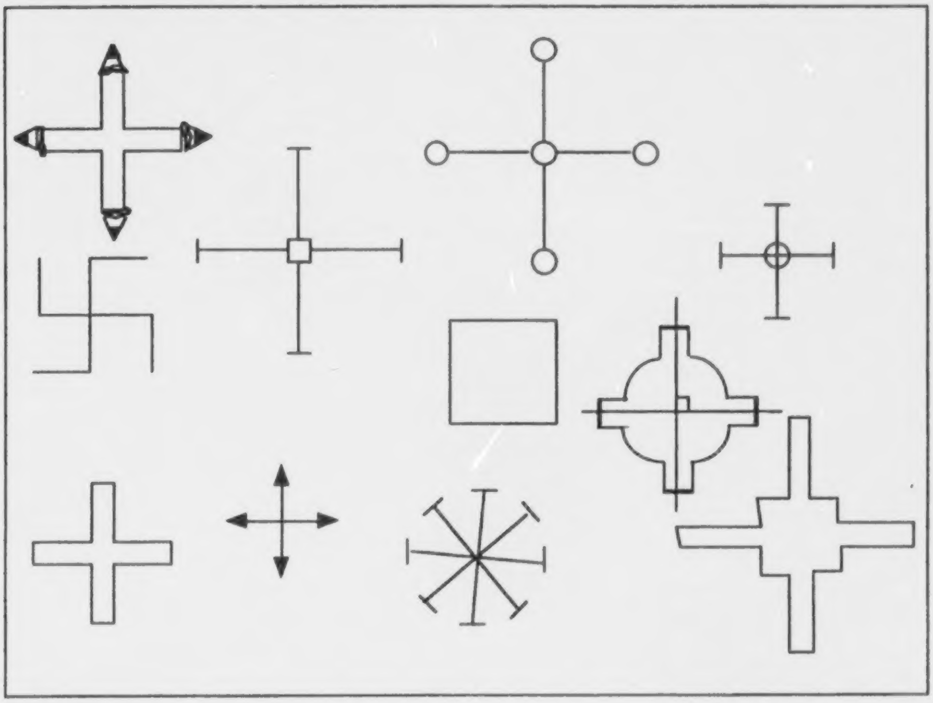


**Fig 1: Teacher A: Observation 4, 11/11/99** **Board 1: at end of lesson**

and here is an extract from the conversation at the start of the lesson:

*Teacher A: Observation 4, 11/11/99* ( - means Teacher A is speaking.)

- If you get your homework out in front of you please. What was it? Just order 4 was it? (Hands go up. 3 board pens are given out: pupils draw a shape and pass the pen (see Fig1)). Look at the ones on the board, are they the same? They might have things that are slightly different.
Pupil: That's a Nazi symbol.
- Given that it's Armistice Day, let's think of it as a Tibetan peace symbol because it is that aswell. Comments?
Pupil: They're all crosses in one way or another.
- All these according to you are order 4. What's similar about them?
Pupil: 4 lines - across, down or diagonal.
Pupil: All got 4 lines.
Pupil: Put in a mirror - you see a reflection.
- Any that haven't got reflection? If we've got order 4 rotation, would we have lines of symmetry? ... Is that always true?
Pupil: All the shapes that have order 4 have 4 lines of symmetry.

> Pupil: I've found one that hasn't.
> - Any on the board?
> Pupil: That one, the peace symbol.
> Pupil: If you put a mirror, you could join the lines up and it would have 4 lines of symmetry.

The collecting of homework examples provides a rich collection of images for which Teacher A invites the consideration of same/different. In his second interview he talks about using these questions:

> If there's more than one thing you're going to do comparisons and you're going to talk about what's the same and what's different. So, I haven't had the feeling that it's something artificial, it just seems natural to the whole sort of discussion of 'can you explain it?', 'what's going on here?', 'why do you think that?' (9/12/99).

The teacher's use of same/different is contingent upon the pupils' observations: *All got 4 lines* and *...you see a reflection*. Teacher A mirrors back a question offering: *If we've got order 4 rotation, would we have lines of symmetry?* to which a pupil makes a conjecture: *All the shapes that have order 4 have 4 lines of symmetry*, and another pupil gives a counterexample: *That one, the peace symbol*. The teacher's use of same/different to support mathematical argument is more sophisticated but nevertheless contingent upon the pupils' comments. In the interaction space is created for the pupils to make connections *(Principle 1)*. They are exploring whether things always work and if not under what conditions they hold. The teacher had not expected the lesson to go in the direction which it did. His planning had been on the level of structure rather than detailed content ie how to work with the range of homework the pupils would bring with them.

In the second whole day meeting of the group of teachers and researchers, having watched two contrasting videotape extracts and worked on discussing the samenesses and differences between them *(Principle 3)*, we started to talk about the two teachers being'fuelled by the kids' and worked on observing teaching strategies which supported this. In this way a teaching 'theory', developed out of same/different, allows the possibility for extending our range of practices as teachers, which then feeds back into how we think about the theory. It is also clear that the language we are developing is only useful for the group of us who went on this particular journey *(Principle 4)*.

### Pupils' developing the use of same/different to generate questions

A second teacher, Teacher D, worked on four problems (of which the first three are discussed here) throughout the first term of the school year, September to December, 1999. The following extracts are taken from the written work of the six pupils who are being interviewed from this year 7 class. What strikes us is how the pupils are now using same/different to generate questions and foci for taking forward their thinking on the problems (see Problem 3) after initially making simple observations (Problem 1). The six pupils are across the spread of achievement within the class and, as a group, indicated by standardised school tests at the start of the year, are below national averages. We have edited the pupils' writing for spelling.

*Problem 1: 1089:* A number trick where students are initially invited to take any three digit number where the units digit is smaller than the hundreds digit, eg 453 and subtract from it the reversed number, 354. Reversing the answer to this calculation and adding gives 1089. What happens with four digits? five digits? The common boards are used for collecting, checking and analysing results.

> Pupil 1: I notice that the same numbers keep appearing in the addition part.
> Pupil 3: It is impossible to find any other answer than 1089. That is my opinion.
> Pupil 4: This didn't work. When I did the first two numbers (897 - 798 = 99) the answer was less than 100. If I do the two numbers and the answer is less than 100 it always adds up to 198.

*Problem 2: Polyominoes:* How many shapes can be made from squares of the same size joined edge to edge and corner to corner as the number of squares increases? The common boards were used to share methods for being organised about knowing how many there were.

> Pupil 2: If you are very organised you will get them all. By being organised we are rotating one square and starting from the highest point. We don't allow duplicates or if you can reflect it.
> Pupil 5: How many different shapes can you get with 4 squares?
> Pupil 6: I am being organised by putting all of the hexominoes in the right order and making sure that there are no hexominoes that are the same.

*Problem 3: Functions and graphs*: A game was played where a rule or function (of the form N goes to 2N +1) was guessed and pupils described how they had found the rule in a range of different ways on the common board. The invitation was to plot such rules as graphs and explore.

> Pupil 1: I have looked at various rules that got the same answers and I worked out they are in fact the same rules, just written differently. We found that (2N + 5) x 2 is the same rule as 4N + 10. So, if we had (3N + 4) x 5 it is the same as 15N + 20. But is $(N + 3)^2$ the same as $N^2 + 3^2$?
> Pupil 2: On the graphs there are two types of lines curved and straight. There are different rules to make either line. I have worked out both types of rules. Here are two conjectures.'If you have N x N in the rule, it will be curved' and 'If it is straight then it doesn't have N x N in the rule'.
> Pupil 3: If we try different rules will the graph still work? Can we be organised in finding all the rules?
> Pupil 4: A question I am going to do next is 'Is it always a curved line with two n's?'
> Pupil 5: From what we were talking about I have learnt that not all the graphs are straight and not all the graphs are curved.
> Pupil 6: When it goes up in curved lines, why isn't it going up in the tables?

During the term we observed pupils; using their powers of discrimination simply to describe the problem contexts (Problem 1, pupils 1, 3, Problem 3, pupil 5), using descriptions linked to action (Problem 2, pupils 2, 6) and using more complex formulations of questions which themselves are generated from the pupils own use of same/different (Problem 2, pupil 5, Problem 3, pupils 1, 2, 3, 4, 6). Although, as in Teacher A's lesson the pupils' thinking is supported contingently, we had not anticipated that the form of the pupils' questions would use samenesses and differences so explicitly. This seemed strong evidence for same/different supporting pupils' asking their own questions and developing as mathematicians as they need to describe and justify the structures they explore.

## Further thoughts

One aspect of this work in progress that we still find surprising is the way that the teacher's use of same/different in their planning and action has reproduced within the ways that the pupils do mathematics. We were being explicit about them 'becoming mathematicians', not about the use of same/different. Is this an example of communication re-producing itself? (Steinbring, 1999, p.42). Teacher A's focus was certainly not on what he wanted from the activity. This seems to be an example of *authentic mathematical interaction* which Steinbring (1999) describes as follows:

> Instead of pushing ahead her own ideas, the teacher places the students' solution-strategies into the foreground of the communication process (p.53).

Instead of learning what is in the teacher's mind the pupils are learning mathematics through communicating mathematically and this is being done 'naturally' through same/different. We are interested in whether this awareness can extend our practice.

1 'Developing algebraic activity in a 'community of inquirers'' Economic and Social Research Council (ESRC) project reference R000223044, Laurinda Brown, Ros Sutherland, Jan Winter, Alf Coles. Contact: Laurinda.Brown@bris.ac.uk or Laurinda Brown, University of Bristol, Graduate School of Education, 35 Berkeley Square, Bristol BS8 1JA, UK

## References

Bateson, G. 1978, 'Afterword'. In J. Brockman (Ed.), *About Bateson*, London: Wildwood House, pp235- 247

Brown, L., Coles, A. 1997, 'The Story of Sarah: Seeing the General in the Particular?'. *Proceedings of the Twenty-first Annual Conference of the International Group for the Psychology of Mathematics Education*, Lahti, 2, 113-120

Brown, L., Coles, A. 1999, 'Needing to use algebra - a case study'. *Proceedings of the Twenty-third Annual Conference of the International Group for the Psychology of Mathematics Education*, Haifa, 2, 153-160

Engeström, Y., Cole, M. 1997, 'Situated cognition in search of an agenda'. In D. Kirshner & J. Whitson (Eds), *Situated cognition: social, semiotic, and psychological perspectives*, New Jersey: Lawrence Erlbaum Associates, pp301- 309

Hannula, M. 1998, 'The Case of Rita: 'Maybe I started to like math more''. *Proceedings of the Twenty Second Annual Conference of the International Group for the Psychology of Mathematics Education*, Stellenbosch, 3, 33-40

Reid, D. 1996, 'Enactivism as a methodology'. *Proceedings of the Twentieth Annual Conference of the International Group for the Psychology of Mathematics Education*, Valencia, 4, 203-209

St. Julien, J. 1997, 'Explaining learning: the research trajectory of situated cognition and the implications of connectionism'. In D. Kirshner & J. Whitson (Eds), *Situated cognition: social, semiotic, and psychological perspectives*, New Jersey: Lawrence Erlbaum Associates, pp261- 279

Skemp, R. 1986 (2nd edition), *The psychology of learning mathematics*, Middlesex: Penguin Books

Steinbring, H. 1999, 'Reconstructing the mathematical and social discourse - aspects of an epistemology-based interaction research'. *Proceedings of the Twenty-third Annual Conference of the International Group for the Psychology of Mathematics Education*, Haifa, 1, 40- 55

Sutherland, R. 1991, 'Some unanswered research questions on the teaching and learning of algebra', *For the Learning of Mathematics*, 11-3, 40-46

Varela, F. 1999, *Ethical know-how: action, wisdom, and cognition*, Stanford: Stanford Uni. Press

Vosniadou, S., Ortony, A. (Eds.), 1989, *Similarity and Analogical Reasoning*, New York: CUP

# THE EFFECT OF SOME CLASSROOM FACTORS ON GRADE 3 PUPIL GAINS IN THE LEVERHULME NUMERACY RESEARCH PROGRAMME

Margaret Brown, Hazel Denvir, Valerie Rhodes, Mike Askew, Dylan Wiliam and Esther Ranson

King's College, University of London

*Abstract: Mean class gains which English Grade 3 pupils in 73 classes made in an assessment of numeracy have been matched against a range of data, including teacher questionnaire, classroom observation and teacher interviews, to examine the effect of whole class teaching, use of calculators and homework. The results confirm much existing research in demonstrating that these variables fail to explain the variation in mean gains between different classes . This suggests some doubts about the empirical basis of aspects of current government policy in England.*

## BACKGROUND AND AIMS

The five year Leverhulme Numeracy Research Programme (1997-2002) focuses on attainment in numeracy in English primary schools (ages 5-11 years) and on ways of improving numeracy standards. The Programme contains a Core Project and five Focus Projects in specific areas. The Core Project, which provides the data for this paper, sets out to obtain large scale longitudinal value-added data on numeracy in order to both inform knowledge about the progression in pupils' learning of numeracy throughout primary school and assess relative contributions to gains in numeracy of the different factors investigated in the programme.

Shortly after the beginning of this research programme the British government outlined proposals for a National Numeracy Strategy to be implemented in all schools in 1999/2000 (DfEE 1998).

Features of the National Numeracy Strategy include:

- recommended national objectives for each week of each year
- much greater concentration on mental calculation
- less calculator use, and none before grade 4
- more direct teaching of the whole class together
- more homework
- a systematic programme of training for all primary teachers.

This paper reports results from the first year of the Leverhulme project, examining the contribution to effectiveness in teaching numeracy of some features of the

National Numeracy Strategy: calculator use, whole class teaching, homework. Relevant literature will be discussed in the appropriate sections below.

## COLLECTION AND ANALYSIS OF DATA

Ten schools were selected in each of four varied Local Education Authorities (LEAs). The ten schools were selected to represent the variations within each authority on the basis of the school size, the socio-economic nature and mathematical level of pupil intake, religious affiliation, and value-added performance. The progress of two cohorts is being monitored over 4/5 years.

Data is collected in six ways:

- Pupils are assessed twice in each school year (October and June), except at the Reception stage, using a linked series of numeracy assessments.
- Pupil data is collected detailing, for example, postcode of home address, details of performance in baseline and national tests, etc.
- Each teacher completes a questionnaire, which provides data about their beliefs and practices in teaching mathematics, and teacher characteristics such as age, gender, qualifications.
- Each class is visited for one mathematics lesson each year: the visiting researcher observes and records notes about the lesson.
- The class teacher is interviewed by the researcher who has observed the lesson, exploring the teacher's views on the lesson, the pupils and the mathematics as well as teacher characteristics and preferences.
- Both the headteacher and mathematics co-ordinator in each school were interviewed in the first year of the study, with updates in subsequent years.

This paper reports some of the results of the first year of the Programme (1997-8) in which only cohort 2 (Grade 3) children were assessed. Scripts were returned by 1723 pupils in the Autumn term and 1655 in June. Results reported in this paper are for the 1467 pupils who could be matched for both administrations of the test, but excluding eight pupils in one mixed age class whose school experience was severely disrupted by the absence of a permanent class teacher.

A series of six pencil-and-paper tests have been designed for pupils in each year, Kindergarten to Grade 5 inclusive. The assessments were adapted from previous work undertaken at King's and used in the *Effective Teachers of Numeracy* project (Askew et al., 1997) which were, in turn, based on an orally administered diagnostic numeracy test for whole classes aged 7-11 (Denvir and Brown, 1987).

An attempt was made to include contextual as well as purely numeric items, application and conceptual as well as procedural skills. In order to focus on the degree of sophistication of children's mental strategies rather than routine written procedures or primitive counting strategies, the assessment consists chiefly of open response orally administered items with a controlled response time. The same test was used in November and June.

To determine the "value added" for children at the extremes of attainment the items had a near uniform facility distribution. In order to smooth out the data so that the expected mean gains in each class were the same for the whole range of mean scores, the data was adjusted using Tukey's lambda transformation (Tukey, 1977).

The adjusted class gains were related to the data from the teacher questionnaire, in some cases validated with reference to observation and interviews. Items in this questionnaire were adapted, with permission, from TIMSS (Mullis et al., 1997).

## FACTORS RELATING TO MEAN CLASS GAINS: RESULTS AND DISCUSSION

Results are summarised in Table 1 in the appendix, which shows effect sizes for the selected factors, and their significance, in increasing order of magnitude. It should be noted that only one result is significant and then only at a very low significance level. It is important therefore not to read too much into the ordering of the factors.

In interpreting these data it should be noted that the numbers of teachers in some categories were small, and that the total sample was around 64 teachers in most cases, since 9 of the 73 teachers did not complete the questionnaire. It is also important to note that statistical associations do not imply causality.

### 1. Frequency of whole class teaching

Over the last few years there has been pressure on primary school teachers to devote more time to direct whole class teaching in mathematics. The National Numeracy Strategy (DfEE 1999) claims that "Better numeracy standards occur when teachers devote a high proportion of lesson time to direct teaching of whole classes and groups." There have already been changes: results for teachers in the TIMSS study (Mullis et al. 1997) in 1994 show that only 11% of English primary teachers in Grade 4 teach directly to the whole class every day. This was a much smaller proportion than that of teachers in almost any other country The same question asked, in our study, of Grade 3 teachers, only three years later, gives 52% who teach the whole class directly every day. Our observation and interview data suggest that the proportion was probably nearer 60% by the Spring Term in 1998, 18 months before the introduction of the Numeracy Strategy and before the Task Force had issued their final report.

In spite of the claims of the Task Force that 'there is support in the research' for 'an association between more successful teaching of numeracy and a higher proportion of whole class teaching' (DfEE, 1998, p.19), the evidence is not unambiguous. Many medium and large-scale studies show no significant effect (Aitken, Bennett & Hesketh; Burstein, 1992; Askew et al., 1997; Creemers,1997).

In other large scale statistical studies there has been a positive correlation between whole-class teaching and attainment (Galton and Simon (Eds), 1980; Good, Grouws

& Ebermeier, 1983; Brophy & Good, 1986). However, noting that in individual cases particularly poor results have also been associated with whole class styles, investigators have cited evidence for the quality of teacher-pupil interaction being a much more important factor than class organisation (Good, Grouws & Ebermeier, 1983; Good and Biddle, 1988; Galton, 1995). These authors suggest that a whole class format may make better use of high quality teaching, but may equally increase the negative effect of lower quality interaction. Peterson(1979), in a review of mathematical learning studies, found that with the more direct approaches of traditional whole class teaching, students tended to perform slightly better on achievement tests (although the effect sizes were small). However they performed worse on tests of more abstract thinking, such as creativity and problem-solving.

The item in the questionnaire relating to whole class teaching was identical to that in the TIMSS questionnaire in order to maintain comparability across time. However it was not very satisfactory in that it asked only for the frequency of whole class teaching in mathematics and not the amount. In order to check the data collected in the questionnaire we matched the questionnaire responses with interview and classroom observation data, bearing in mind that most teachers agreed that they did more class teaching than usual in the observed lesson, while a few did less. This additional data either confirmed each teacher's response on the questionnaire or showed up inconsistencies: those 12 out of 64 teachers whose replies were unclear or inconsistent were removed from the analysis. In checking through observation and interview data we also examined whether teachers who said that they were whole class teaching every day or nearly every day were doing so for a substantial time (ie more than 25% of lessons); this was almost always the case, and where there was any uncertainty the teacher was removed form the final sample.

Further analysis of our qualitative data reveals that "individualised" work in the sense that had been observed and described for example by Galton & Simon (1980) did not occur in any of the 74 mathematics classes. Children did not work on their own through books or schemes doing work which was unrelated to the work that others were doing. The only Grade 3 teachers who did not class teach for a substantial period in every mathematics lesson described their practice as introducing a new topic to the whole class in about one lesson a week and sometimes coming back to the whole class at the end of the topic. In the intervening lessons children in general worked in ability groups on differentiated tasks, often with the teacher giving direct teaching to particular groups.

The detailed results showed that the amount of whole class teaching had no significant association with class mean gains in the assessment, although those who taught the whole class less frequently had slightly higher gains. This is not inconsistent with other results reported above. However it should be noted that this lack of effect may be partly due to the fact that the variety of teaching organisation was by early 1998 quite small and much reduced from what has previously been observed in English primary mathematics classrooms.

The results suggest that the Numeracy Strategy may over-emphasise the effectiveness of whole class teaching, and that the proportion of whole class teaching had already increased substantially well before the Strategy was implemented.

### 2. Frequency and nature of use of calculators

Teachers are being advised that in primary schools calculators should not be used as a calculating tool; in particular the National Numeracy Strategy recommends that calculators should not in general be used with pupils below Year 5 (DfEE, 1998).

Research on the effect of calculator use is inconclusive. Ruthven (1997) in a careful review of research on the effect of calculators on attainment in numeracy concludes that in primary schools:

> 'the degree of calculator use remains modest in most schools and by most pupils…however tempting it might be to cast the calculator as scapegoat for disappointing mathematical performance at primary level, the available evidence provides scant support for this position...' (Ruthven, 1997, p18)

In our study teachers were asked to indicate how often they used calculators in their teaching and how often children had access to them. The results show that the frequency of use of calculators in mathematics teaching, whether almost every day, once or twice a week or less frequently, was not significantly associated with the size of mean class gains. Neither was the accessibility of calculators for children; although access to calculators on most days was slightly more likely to be associated with lower gains and rare access with higher gains, there was some inconsistency for the intermediate cases.

Teachers were also asked to indicate the purpose of calculator use in class. The frequency with which classes used calculators to learn to use them efficiently, for checking answers, routine computation, developing number concepts or solving complex problems did not have a significant effect on class gains. There was however some tendency for more frequent use for conceptual purposes or learning to use a calculator to be associated with higher gains.

This suggests that the National Numeracy Strategy is over-cautious about its warnings about over-use.

### 3. Frequency and nature of homework

Whilst the value of homework at secondary level is well established (Hallam and Cowan,1998) the situation at primary level is less clear. A recent study of Grade 5 pupils indicated that frequent regular homework in mathenatics was associated with less progress than infrequent homework(Farrow, Tymms and Henderson, 1999).

In our study teachers were asked to indicate how often they set homework for their class. Although the children in classes of teachers who set homework once or twice a week had very slightly higher mean class gains in attainment than teachers who set homework once a week, the difference was not significant.

Teachers were also asked to indicate the purpose of the homework. Doing more class work, finishing of work set in class and learning tables did not have a significant effect on class gains. Classes who were not set problem solving activities or investigations did slightly better than classes who were although again the differences were not significant.

In contrast, classes where homework was focused on family activities e.g. the IMPACT scheme (Merttens & Vass, 1993) did significantly better than those which did not. However it should be noted that the number of classes was small, only 5 out of 64 in all. These findings are not consistent with the latest Government advice and support those of Farrow et al (1999) that more homework, unless it is the family activity approach of the IMPACT scheme, is not necessarily better.

## CONCLUSIONS AND ALTERNATIVE DIRECTIONS

Differences in classroom practices identified from the teachers' questionnaire, in some cases confirmed against classroom observations and interviews, were on the whole unrelated to the mean class gains that pupils made in the assessments. We believe that determining 'what works' is not an easy task since most easily measurable and apparently 'obvious' factors seem in our study not to have any significant association with higher gains in numeracy. This failure indicates equally that simple recipes are likely to have little effect.

As we and many researchers both in the UK and in other countries have suggested, we believe that the differences between effective and less effective teachers are more subtle than this, and relate rather to characteristics of teachers' integrated knowledge, beliefs and practices in mathematics teaching (e.g. Askew et al., 1997).

Our current work, consequently, focuses on developing a theoretical framework for examining interactions in primary mathematics classrooms. This is reported in a parellel paper presented at this conference (Askew et al., 2000)

## REFERENCES

Aitken, M., Bennett, N. and Hesketh, J. (1981) Teaching Styles and Pupil Progress: a re-analysis. *British Journal of Educational Psychology, 51*, 170-186.

Askew, M., Brown, M., Rhodes, V., Wiliam, D. and Johnson, D. (1997) *Effective Teachers of Numeracy.* London: King's College London.

Askew, M. et al. (2000) Describing primary mathematics lessons observed in the Leverhulme Numeracy Research Programme: a qualitative framework. *Proceedings of PME 24.* Hiroshima: University of Hiroshima.

Brophy, J. E. & Good, T. L. (1986) Teacher behaviour and student achievement. In M. Wittrock (Ed.) *Handbook of Research on Teaching.* New York: Macmillan.

Burstein, L (1992) *The IEA Study of Mathematics III: student growth and classroom processes.* Oxford: Pergamon Press.

Creemers, B. (1997) *Effective schools and effective teachers: an international perspective.* Warwick, UK: Centre for Research in Elementary and Primary Education

Denvir, B. & Brown., M. (1987). The feasibility of class administered diagnostic assessment in primary mathematics. *Educational Research,* 29 (2), 95-107.

Department for Education and Employment (DfEE) (1998) *The Implementation of the National Numeracy Strategy: The final report of the Numeracy Task Force.* London: DfEE.

Department for Education and Employment (DfEE) (1999) *The National Numeracy Strategy.* London: DfEE.

Farrow, S., Tymms, P. & Henderson, B. (1999) Homework and attainment in primary schools. *British Educational Research Journal,* 25 (3), 323-341.

Galton, M., and Simon, B. (Eds) (1980) *Progress and Performance in the Primary Classroom.* London: Routledge.

Galton, M. (1995) *Crisis in the Primary School.* London: Routledge.

Good, T.L., Grouws, D.A., & Ebmeier, H. (1983) *Active Mathematics Teaching.* New York: Longman.

Good, T.L. & Biddle, B.J. (1988) Research and the improvement of mathematics instruction: the need for observational resources. In D.A. Grouws & T.J. Cooney (Eds.), *Perspectives on Research on Effective Mathematics Teaching,*(p.114-142). Reston, Va.: NCTM/Lawrence Erlbaum.

Hallam, S. & Cowan, R. (1998) *Is homework important for increasing educational attainment?* Unpublished paper, Institute of Education, University of London.

Merttens, R. & Vass, J. (Eds.) (1993) *Partnership in Maths: Parents and schools: the IMPACT Project.* London: Falmer.

Mullis, I. V. S, Martin, M. O., Beaton, A. E., Gonzalez, E. J., Kelly, D. L. & Smith, T. A. (1997) *Mathematics Achievement in the Primary School Years: IEA's Third Mathematics and Science Study.* Chestnut Hill, Mass: Boston College.

Peterson, P. L. (1979) Direct instruction reconsidered. In P. L. Peterson & H. J. Walberg (Eds.) *Research on Teaching: Concepts, findings and implications* (pp 57-69). Berkeley, CA: McCutchan Publishing Corporation.

Ruthven, K. (1997) *The Use of Calculators at Key Stages 1-3 (Discussion Paper no. 9).* London: School Curriculum and Assessment Authority.

Tukey, J.W. (1977). *Exploratory Data Analysis.* Reading, MA: Addison-Wesley.

## APPENDIX

**Table 1: Effect sizes of factors, significance levels and numbers of classes (n).**

| *Factors with effect size less than 0.2 (all non-significant, i.e. p>0.1)* | | | |
|---|---|---|---|
| Homework set less than once a week (v. once or more) | | | |
| Homework set to finish off work set in class | | | |
| Homework set to continue classwork | | | |
| Homework set to learn multiplication tables | | | |
| Access to calculators once or twice a week | | | |
| Access to calculators once or twice a month | | | |
| How often maths teaching involves the use of calculators | | | |
| | | | |
| ***Factors with effect size greater than 0.2*** | **Effect size** | **p** | **n** |
| Calculators frequently used for checking answers | -0.21 | n.s. | 17 |
| Whole class teaching once or twice a week (v. most days) | 0.22 | n.s. | 21 |
| Calculators sometimes used for routine computation | 0.24 | n.s. | 18 |
| Pupils rarely have access to calculators | 0.27 | n.s. | 16 |
| Calculators rarely used to develop number concepts | -0.28 | n.s. | 20 |
| Calculators rarely used for solving complex problems | -0.31 | n.s. | 17 |
| Access to calculators most days | -0.33 | n.s. | 10 |
| Calculators frequently used to learn to use a calculator efficiently | 0.36 | n.s. | 4 |
| Homework set on problems or investigations | -0.36 | n.s. | 22 |
| Calculators rarely used for becoming familiar with number operations | -0.41 | n.s. | 16 |
| Homework set on family activities e.g. IMPACT | 0.89 | p=0.07 | 5 |

# "AUTOMATISM" IN FINDING A "SOLUTION" AMONG JUNIOR HIGH SCHOOL STUDENTS – A COMPARATIVE STUDY

Gildo Luís Bulafo

Universidade Pedagógica, Beira – Mozambique

*Ilany & Shmueli (1998) studied how secondary school pupils and teachers in Israel solved a given equation. The objective was to find out whether the participants found a solution through a routine algorithmic process or through various other strategies. From their study, Ilany & Shmueli concluded that the students mostly resorted to extracting and substituting the variable (routine), while the teachers employed predominantly direct calculation. In a similar study conducted in Mozambique the same tendency was found. However, students in Mozambique used two strategies not found by Ilany & Shmueli: (1) direct calculation followed by extraction of the variable; (2) changing an expression into an equation and solving it. These strategies indicate that the students are convinced that the final answer always must give a specific value to an unknown. This paper will attempt to analyse some of these differing results, taking the respective systems of education into account.*

## 1. INTRODUCTION

At the Beira Campus of the Pedagogical University, Mozambique, Mathematics teachers for secondary schools are trained through the BLEM programme (Bachelor & Licenciature in Mathematics Education). In the discipline of Didactics of Mathematics aspects of the teaching and learning process are studied through various topics of "School Mathematics". The purpose of these studies is: a) the integration of trainee-teachers in the complexity of the process of teaching and learning; b) the discovery of pupils' abilities and their difficulties with the learning of Mathematics; c) to contribute to a change of the teachers' attitudes: away from the idea that only the teacher knows the rules and the solutions and towards more openness for the pupils' ideas giving them more opportunities to participate actively in their own learning. One of the topics which was selected was "School Algebra". The researcher (lecturer of Didactics) had already observed some of the pupils' difficulties when he was a secondary teacher himself. This paper presents a comparative study of strategies used in the solving of elementary equation by pupils and teachers in Israel and Mozambique. While in Israel secondary education starts with the 7th grade, in Mozambique it starts with the 8th grade; we therefore assume a certain correspondence in level among the two grades.

## 2. METHOD

As a way of initiating the study of questions related to "School Algebra", 2nd year students of the BLEM programme were asked to conduct an inquiry similar to the one described by Ilany & Shmueli (1998), with pupils and teachers in Beira city. At the same time as the results were organised and analyzed, the students studied the Küchemann (1981) paper on different uses of letters and several school books (Mozambican and from some other countries). So conducting the enquiry, organizing and discussing the data was part of the professional training of the trainee-teachers.

### 2.1 The Ilany & Shmueli study

#### a) The questionnaire

Ilany & Shmueli's study in Israel focused on 92 pupils from the 7th grade (30), from the 8th grade (30) and from the 9th grade (32), as well as on 12 junior high school teachers for these grades. The participants were asked to solve the following tasks:

1. If $3+a=8$ then $3+a+5=$
2. If $3d=6$ then $18d+9=$
3. If $e+f=8$ then $e+f+g=$
4. If $5(2a+1) = 10$ then $\frac{2a+1}{2} =$
5. If $x+y=z$ then $x+y+z=$

#### b) Criteria for analysis

Ilany & Shmueli used the following categories:

*1. Direct calculation – global structural perspective*

For the participants the unknown is not just a letter but an expression.

E.g. in the first problem, they saw that $3 + a$ equals 8 and recognized the same expression in $3 + a + 5$; therefore they substituted $3 + a$ by 8, obtaining $8 + 5$ or 13.

*2. Substitution in the "reverse direction" – global structural perspective*

In order to find out how much $x + y + z$ is, when $x + y = z$, the value of z is exchanged by $x + y$ in the 1st expression, instead of substituting $x + y$ by z. The result is: $x + y + z = x + y + \underline{x + y} = 2(x + y) = 2x + 2y$

*3. Extraction of the variable and its substitution – routine algorithmic perspective*

To find out how much is $3 + a + 5$, when $3 + a = 8$, the participant calculated the value of "a" in $3 + a = 8$ and substituted it in the 2nd expression. Thus: $3 + a + 5 = 3 + 5 + 5 = 13$

*4. Interim method –partial structural perspective*

The unknown may not only be a letter or a whole expression, but also a part. To find out how much is $\frac{2a + 1}{2}$, when $5(2a + 1) = 10$, the participants calculated $2a = 1$ and then substituted this and obtained $\frac{2a + 1}{2} = \frac{2}{2} = 1$

5. *Extraction of the variable only without substitution - partial algorithmic perspective*

The value of the letter was extracted in the first equation and not substituted in the 2nd expression.

6. *Substitution by any number*

To find out how much is $x + y + z$, when knowing that $x + y = z$, the participants substituted the letters by arbitrary numbers.

7. *Both expressions have the same solution*

To find out how much is $3 + a + 5$, when $3 + a = 8$, the participant thought that:

$3 + a = 8$ therefore $a = 5$ and $3 + a + 5 = 8$, therefore $a = 0$.

### 2.2 The study by the BLEM students (trainee-teachers)

For this study the same questionnaire and criteria for analysis employed by Ilany et al. were used. In the questionnaire one modification was introduced: the 3rd question was substituted by another one, because we considered it smiliar to the 5th one. The problem was substituted by a problem taken from Lemut & Greco (1998), namely:

> "Mark's father is 38, Mark is 6. In how many years
> Mark's father's age will be three times Mark's age ?"

The results of this problem will not be discussed in this paper as it has a different structure from those set up by Ilany & Shmueli.

In a first phase, the lecturers of Didactics of Mathematics asked the 9 trainees to answer the questionnaire. The results were analyzed and discussed during the Didactics seminars. Then the BLEM students asked seven Mathematics teachers at two Junior High Schools in Beira to respond to the questionnaire. The same questionnaire was responded by a total of 318 pupils (110 of grade 8, 139 of grade 9, 69 of grade 10). Apart from these participants, the author conducted the same inquiry with 41 pupils of grade 9, in the rural town of Marromeu, located 400 km from Beira.

The inquiry in the Beira schools was conducted by the trainee-teachers, including the organization and discussion of the data, as part of their studies. The results were presented and discussed in the Didactics of Mathematics seminars. The analysis of the answers of teachers and pupils, according to the criteria adopted by Ilany & Shmueli and complemented by Küchemann (1981) proved to be a difficult task. The tables of results had to be changed four times, according to improved classification of the responses. The results of the Beira teachers and pupils were presented and discussed with the teachers by the lecturers and students of the BLEM.

## 3. RESULTS OF PUPILS

Table 1 shows the results obtained from the pupils in Israel and Mozambique. The results allow a comparison of the use rate of the letter among the different grades in Mozambique and in Israel, and also of the pupils of the two countries.

*Table 1: Results of pupils in Israel (I) and in Mozambique (M), in percentages*

| *Grade, Country, N* | 7th I | 8th M | 8th I | 9th M | 9th I | 10t M | tot I | tot M |
|---|---|---|---|---|---|---|---|---|
| *Sulution methods* | **30** | **110** | **30** | **180** | **32** | **69** | **92** | **359** |
| Extracting and substituting the variable – routine algorithm | 43% | 26% | 45% | 45% | 46% | 52% | 45,5 % | 41% |
| Direct calculation – structural perspective | 15% | 3% | 31% | 8% | 41% | 13% | 29% | 7% |
| Interim method – substituting part of expression | – | 0,5% | 1% | 2% | 3% | 3% | 1% | 2% |
| Substituting any numbers | 28% | 32% | 12% | 13% | 4% | 10% | 14% | 1% |
| Extracting variable only – no substitution | 2% | 12% | 1% | 12% | 1% | 3% | 1% | 10% |
| "Reverse substitution" | – | 0,5% | 2% | 3% | – | 2% | 0,5% | 2% |
| Same solution for both expressions | 1% | – | 2% | – | – | 0,6% | 0,5% | 0,1% |
| Direct calculation and then extract the variable (strong routine) | – | 0,5% | – | – | – | 3% | – | 1% |
| Changing of the 2nd expression into an equation and solving it | – | 17% | – | 13% | – | 6% | – | 13% |
| No response | 11% | 8% | 6% | 4% | 5% | 8% | 8% | 5% |

The table shows:

- Apart from the seven strategies met by Ilany & Shmueli, two new strategies were found among the Mozambican pupils.

## 4. DISCUSSION OF PUPILS' RESULTS

### 4.1 The most used strategies

- The table shows that the strategy of *extraction of the variable and its substitution* is most used by the Israeli pupils with 45,5% and also by the Mozambican pupils with 41%. While in Israel this strategy is used equally among the different grades, in Mozambique it shows a gradual increase, from 26% in the 8th grade to 52% in the 10th grade.
- The second most used strategy in Israel is *direct calculation* with 29%, which is the fifth most used among the Mozambicans with only 7%. In both countries the rates are gradually increasing with the grade level, but in Mozambique the percentage of usage is much lower and starts at only 3%. In the 10th grade in Mozambique it reaches 13%, which is still less than the Israeli percentage for the 7th grade (15%).

*Comments*

The rates of usage of the routine algorithmic strategy are almost constant for Israeli pupils, while they are gradually incleasing for Mozambican pupils. The explanation may be as follows. In the Israeli 7th grade a majority of nearly 60% of the pupils uses correct strategies (algorithmic routine, direct calculation or interim method). In the Mozambican 8th grade only 30% of the pupils uses a correct method. With increasing grade level, in both countries, the usage of incorrect strategies decreases – probably as a result of teaching and learning. This is reflected by an improvement in the use of variables for Israeli pupils and an increase in the use of the direct calculation method. For Mozambican pupils the improvement corresponds to an important increase in the use of the algorithmic routine and a very modest increase in the use of the direct method. In other words, the higher Mozambican percentage for the routine method (52% in the 10th grade) as compared to the Israeli data is caused by the fact that a rather small percentage of Mozambican pupils passed on to the more advanced direct method. Overall, the use of correct methods increases in both countries: in Israel from 58% in the 7th grade to 90% in the 9th grade; in Mozambique from 30% in the 8th grade to 68% in the 10th grade.

### 4.2 Other strategies

– *Substitution by an arbitrary number* was a much used strategy in Israel (14%) as well in Mozambique (19%). The occurrance of this method decreases with the increase of grade level and experience. For Mozambique, solutions in which it was not clear whether the value of 1 was given to the variable or whether the variable was just ignored as can be seen in the following example, were included:

*Example:* Task 4

The question was: If $5(2a+1) = 10$ then $\frac{2a+1}{2} = ?$

Several pupils wrote: $\frac{2a+1}{2} = \frac{2+1}{2} = \frac{3}{2}$ which may be interpreted either way.

The tendency of using this kind of strategies was already observed by Küchemann (1981) and MacGregor & Stacey (1997). Either the variable $a$ is ignored, because pupils don't know how to handle them, or the pupils establish a correspondence between letters and numbers, giving $a$ the value 1 because it is the first letter of the alphabet.

– *Interim method*: there are few students found in the transition between the extraction–substitution method and direct calculation ("interim method"); in Israel with only 1% as well as in Mozambique with only 2%.

– *Extracting the variable without substitution*: in Mozambique many students (10%) used this method, probably because they did not understand the problem and limited themselves to calculating the value of the unknown through a routine algorithm. In Israel only 1% of the pupils used this method. In the Mozambican 10th grade, the percentage is reduced to 3%, due to more experience.

– The solution methods *"Reverse substitution"* and *the same solution for both expressions* were practically not used in the two countries.

### 4.3 New strategies

Two strategies, which were not part of Ilany & Shmueli's work, were encountered.

*1. Strong routine* – Although at a low rate (0,6%), some Mozambican pupils were thought to have turned to a strategy which may be described as "direct calculation followed by the extraction of the letter" or only "strong routine". This means that the pupils' answers were based on a combination of direct calculation and routine, as one can see in the following examples:

*1st Example:* task 1

The question was, if $3 + a = 8$, then $3 + a + 5 = ?$

The answers were given in the following way:

$$\text{If } 3 + a = 8, \text{ then } 3 + a + 5 = 8 + 5 \rightarrow 8 + a = 8 + 5 \rightarrow a = 5$$

*2nd Example:* Task 4

The question was: If $5(2a+1) = 10$ then $\frac{2a+1}{2} = ?$

The answers were given in the following way:

$$\text{as } 5(2a+1) = 10, \text{ then } \frac{2a+1}{2} = \frac{2}{2}$$

$$\frac{2a+1}{2} = \frac{10}{5} + 2 \rightarrow 2a + 1 - 1 = 2 - 1 \rightarrow 2a = 1 \rightarrow \frac{2a}{2} = \frac{1}{2} \rightarrow a = \frac{1}{2}$$

These two examples show that the pupil received the result in the first step after direct calculation, but the way of presenting the solution in the form of "letter =..." does not allow for understanding that $8 + 5$ in the 1st example or $\frac{2}{2}$ in the 2nd example are already the solutions of the given tasks. What is missing is only a final, simple form of the answer (13 for the 1st example and 1 for the 2nd one). The pupils continued until they found something with the form which they thought that the answer should have.

*2. Transforming the 2nd expression into an equation and solving it* – Among the answers presented by the pupils in Mozambique, 13% of the solutions consisted of attempts to form an equation with the 2nd expression. This strategy, designated "transforming the 2nd expression into an equation and solving it", consists of calculating the value of the letter in the first expression and then imagining that the second expression is also an equation. As the 2nd expression has nothing written to the right side of the equals sign, the pupils transfer one or more numbers from the left side to the right side, obtaining a true equation which may be solved.

*Example:* task 1

If $3 + a = 8$, then $3 + a + 5 = ?$

The second expression is seen as an equation, containing the letter *a* as a specific unknown, whose value must be found. The pupils solved this "equation" in the following way:

If $3 + a = 8$, then $a = 5$, and as $3 + a + 5 =$ , then $a = -5 -3 = -8$

In general the pupils did not notice the contradiction between the different values obtained for the same unknown.

## 5. RESULTS OF TEACHERS AND TRAINEE-TEACHERS

Looking at the table below, the strategies used by the teachers tested by Ilany & Shmueli in Israel and by those tested in Mozambique can be compared.

*Table 2: results of the teachers and trainee-teachers (students of BLEM)*

| *Country (n)*<br>*Strategies* | Israel (12) | Mozam(7) | BLEM (9) |
|---|---|---|---|
| Extraction of the variable and its substitution | 15 % | 36 % | 33 % |
| Direct calculation – global structural perspective | 85 % | 32 % | 61 % |
| Interim method –partial structural perspective | 0 % | 0 % | 6 % |
| Direct calculation followed by extraction of the variable – "*strong routine*" | 0 % | 29 % | 0 % |
| Extraction of the variable only – no substitution | 0 % | 4 % | 0 % |
| Substituting any numbers | 0 % | 0 % | 0 % |
| "Reverse" substitution | 0 % | 0 % | 0 % |
| No response | 0 % | 0 % | 0 % |

### 5.1 Discussion of teachers' data

– The data in the table shows that for every 7th answer given by the Israeli teachers the routine strategy was employed, something also done in every 3rd answer given by the BLEM students (trainee-teachers). The same tendency is also found among the Mozambican teachers.

– Most used among the Israeli teachers was the direct calculation, a similar situation was found among the BLEM group, although the actual percentage differed with 85% among the Israelis and 61% among the BLEM students. The same can, however, not be said for the Mozambican teachers where only every third answer was reached by the use of direct calculation.

– The Israeli teachers either used the routine or the direct calculation, but never any other strategy. Among the BLEM group, 6% were found employing the interim method, a transition between the routine and the direct calculation.

– The situation is radically different among the Mozambican teachers, insofar that they responded in 29% of the tasks using a new strategy, the aforementioned "strong routine". This is a strategy which consists of thinking the solution by using the direct calculation and then feeling compelled to calculate the value of the variable. With 29% using the strong routine and 4% using extraction of the variable only without substitution, it is evident that not only the pupils but also some teachers in Mozambique still have difficulties understanding formulations like "if … then …".

These difficulties may have their origin in the fact that in the schools the strategy of routine is taught as the main strategy for solving equations or problems which have to be translated into an equation before solving them. The tasks used in the questionnaire differ from these school problems: instead of calculating the value of an unknown, pupils and teachers are asked to calculate the value of an expression containing a variables.

## 5.2 Conclusion

It can be said that the schoolteachers tested in Mozambique also use most the method of direct calculation, while the pupils tend to use the routine algorithm of extraction and substitution of the letter. In both cases the succesful usage of these strategies improves with the increasing level of instruction and experience.

Both pupils and teachers in Mozambique found difficulties with tasks containing "if.. ; then ..." as these are not common in schools.

The appearance of two new, incorrect strategies may be due to the difficulties in understanding the given tasks. These difficulties will decrease while the participants gain more experience, either by working more with letters or by the increasing level of instruction.

It is hoped that the trainee-teacher s studying at the Pedagogical University have gained more insight into the particularities of teaching and learning school algebra and the dangers of a teaching approach which emphasizes exclusively the usage of routine algorithms.

## 6. REFERENCES

Ilany, Bat-Sheva & Shmueli, Nurit: 1998, '"Automatism" in finding a "solution" among junior high school students', in Alwyn Olivier & Karen Newstead (Eds), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, University of Stellenbosch, Stellenbosch – South Africa, Vol. 3, p. 56 – 63.

Lemut, E & Greco, S.: 1998, 'Re-starting algebra in high school; the role of Systemic Thinking and of representation Systems Command', in Alwyn Olivier & Karen Newstead(Eds), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, University of Stellenbosch, Stellenbosch – South Africa, Vol. 3, p. 191 – 198.

Kieran, Carolyn: 1981, 'Concepts associated with the eqality symbol', *Educational studies in mathematics* 12, p. 317 – 326.

Küchemann, D.: 1981, 'Algebra', in Kathleen Hart (ed.), *Children's understanding of mathematics: 11 – 16*, John Murray, London – United Kingdom (p. 102 – 119).

# A STUDY OF THE MATHEMATICAL BEHAVIORS OF MATHEMATICIANS: THE ROLE OF METACOGNITION AND MATHEMATICAL INTIMACY IN SOLVING PROBLEMS

Marilyn P. Carlson
Arizona State University

*This study investigated the mathematical behaviors and backgrounds of professional mathematicians. A framework with four broad dimensions was used to analyze the mathematical behaviors of these mathematicians while completing mathematical tasks. The results suggest that affect and control often interacted to influence these subjects' mathematical behaviors. High motivation and interest in specific problems appeared to influence these mathematicians' decision to frequently bring a problem into their consciousness, and to engage cognitively and employ various metacognitive behaviors. When confronting an obstacle they appeared to effectively manage their frustration, with intense cognitive engagement occurring after expressions of tension and frustration.*

Most educators agree that the development of students' problem solving ability is a primary goal of instruction (Lester, 1994). However, in a review of the problem solving literature, Lester (1994) expressed strong concern for the recent decline in problem solving studies. While early studies (1970-1982) in problem solving were concerned with determining the aspects of a task or problem that contributed to its difficulty level (Lester, 1994), more recent research has a different focus, as explained by Lester (1994): "Today there is general agreement that problem difficulty is not so much a function of various task variables as it is of characteristics of the problem solver." Lester's view was echoed by Geiger and Galbraith (1998) who claimed that "it is the relationship between the learner and a problem that is of significance, not the perceived level of the problem as viewed within some hierarchy of abstraction".

In a recent study of expert and novice problem solvers, Geiger and Galbraith (1998) reported that effective implementation of control skills was a key discriminator of problem solving success. Schoenfeld (1992), Vinner (1997) and DeFranco (1998) have also sited control as a misunderstood component of problem solving, with calls for further investigation into the role of control in effective problem solving behavior.

Further, DeBellis and Goldin (1998) claim that affect is a powerful component of one's problem solving behavior. In their 1998 article, they claim that "every individual constructs complex networks of affective pathways, contributing to or detracting from powerful mathematical problem-solving ability". As a result of this recent work the following research questions were formulated for this study:

- How do metacognitive behaviors interact with cognition and affect during problem solving?
- How is control exerted and manifested during problem solving?
- What affective traits, attributes, and experiences contribute to effective problem solving?

The 12 mathematicians in this study were asked to complete four mathematical tasks and were asked to respond to 7 open-ended questions to probe their general problem solving behaviors and the influences that have contributed to their problem solving success. The problems posed were non-routine, and for the purpose of this study, Schoenfeld's (1983) problem definition was assumed:

> *A problem is only a problem if you don't know how to go about solving it. A problem that has no surprises in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an exercise.*

The present study extends the research by examining beliefs, self-knowledge, monitoring and regulation during problem solving, and other metacognitive aspects of problem-solving behavior of *professional mathematicians.*

## Theoretical Framework

The framework for this study extended and combined aspects of previously established frameworks. Schoenfeld's framework classified problem solving behavior and knowledge into four major categories: resources, heuristics, control and beliefs; whereas Lester et al.'s 1985 study considered students' attitudes, beliefs, contextual factors, along with control. Their framework provided a structure that includes four areas where a specific metacognitive decision commonly results in a particular cognitive action. These four areas of problem-solving activities are categorized as Orientation, Organization, Execution, and Verification. Geiger and Galbraith's (1998) framework focussed more heavily on affective components, with Knowledge, Structures, Meta-Processes, and Influences of Beliefs forming the categories of their framework. DeBellis and Goldin's 1998 study focused on affective responses that happen during mathematical problem solving. Her study found that powerful problem solvers possess a willingness to become intimate with mathematics problems, resulting in her classifying mathematical intimacy as a local affect in powerful problem solvers. The framework for the present study contains major components of each of these frameworks, with refinements of control and specific affective dimensions based on past results and these researchers' views regarding the problem solving components that most influence problem solving behaviors.

The broad categories of this framework are resources, control, methods and affective dimensions related to problem solving. For this study control was subdivided into three major dimensions: initial cognitive engagement; cognitive engagement during problem solving; and metacognitive behaviors. Initial cognitive engagement includes activities such as, establishing known information and goals, representation of givens, and organization of information. Cognitive engagement during problem solving includes overt efforts to engage mentally, as characterized by actions of sense-making, construction of logically connected statements, and choosing to bring a problem into one's consciousness. Metacognitive activities include the conscious self-regulation of the problem-solving process, involving activities such as: the evaluation of the efficiency and effectiveness of the selected approach; making and testing conjectures, evaluation of the quality of the cognitive activities, and use of internal dialogue. Other studied aspects of problem solving behavior such as heuristics (Schoenfeld, 1992; Gailbraith, 1999; Geiger & Galbraith, 1998) resources and affect (Debellis, 1995; Lester, 1995) are also investigated and related to aspects of control. The aspects of mathematical behavior investigated in this study follow.

**Resources**

| | |
|---|---|
| RK | Knowledge, facts, and procedures in problem solving success. |
| RU | Conceptual understanding. |
| RC | Collaboration in problem solving success. |
| RT | Technology in problem solving success. |
| RW | Written materials (e.g., texts, journal, Internet) |

**Control**

| | |
|---|---|
| CP | *Initial Cognitive Engagement* |
| CPE | Effort and energy is put forth to read and understand the problem. |
| CPO | Information is organized; |
| CPG | Goals and given are established |
| CPR | Goals and givens are represented using symbols, diagrams, tables. |
| CPS | Strategies and tools are devised, considered and selected. |

| | |
|---|---|
| CE | *Cognitive engagement during problem solving.* |
| CEM | Effort is put forth to stay mentally engaged |
| CEL | Effort is put forth to construct logically connected statements. |
| CEV | Voluntarily brings problem into consciousness and cognitive resources are expended |
| CES | Evidence of sense making/attempts to fit new information with existing schemata |

| | |
|---|---|
| CM | *Metacognitive Behavior s (e.g. self regulation and reflection) Exhibited During the Problem-Solving Process* |
| CMQ | Reflects on the efficiency & effectiveness of cognitive activities |
| CMM | Reflects on the efficiency and effectiveness of the selected methods |
| CMC | Conscious effort to access resources/mathematical knowledge |
| CMG | Generates conjectures |
| CMT | Tests conjectures |
| CMV | Verification of processes and results |
| CMR | Relates problem to parallel problems and experiences |
| CMP | Refines, revises, or abandons plans as a result of what has happened in the solution process |
| CME | Manages emotional responses to the problem solving situation |
| CMU | Uses internal dialogue |

**Methods**

MH The role of heuristics in problem solving success (e.g., work backwards, look for symmetries, substitute numbers, represent situation pictorially or graphically or tabularly, generate conjectures, substitute familiar vocabulary, relax constraints, subdivide the problem, assimilate parts into a whole, alter the given problem so that it is easier, look for a counter-example, investigate boundary values)

**Affective Dimensions Related to Problem Solving**

| | |
|---|---|
| AA | Attitudes |
| AAE | Enjoyment |
| AAM | Motivation |
| AAI | Interest |
| AAU | Perceived usefulness of problem solving activity |
| AB | Beliefs |
| ABC | Self-Confidence |
| ABE | Pride |
| ABP | Persistence facilitates problem solving success |
| ABI | Practice has value |
| ABN | Attempting novel problems has value |
| ABM | Multiple attempts are natural part of the problem solving process |
| AE | Emotions |
| AF | Frustration |
| AA | Anxiety |
| AEJ | Joy, pleasure (aha!) |
| AEI | Impatience and anger |
| AV | Values/Ethics |

AVI Mathematical Intimacy
AVG Mathematical Integrity

## Methods

The subjects were twelve research mathematicians from two large public universities in the Southwest and Western United States. The data sources included: transcription of the subjects' problem solving processes, as verbalized during the completion of four mathematical tasks; and the subjects' verbal responses to seven open-ended questions.

The data for this study was collected during a 90-minute interview. The subjects were allotted 70 minutes to complete the four problems and 20 minutes to respond to the inquiries about their problem solving background and methods. Prior to initiating their work on the four problems, the subjects were directed to "talk through" their thought process while completing the problem. Once each problem had been completed, the subjects were also asked to provide rationale for their attempts and clarification of their approach. The interviewer passively observed, while noting behaviors that emerged during the problem solving process, with special focus on detecting behaviors defined in the theoretical model. As was necessary to optimize the information generated during the interview, the interviewer reminded the subject to verbalize her/his thought processes.

The four problems presented during the interviews required knowledge of elementary content and concepts such as basic geometry, algebra, proportions, and percents. Two of the problems used for this study follow:

1. A square piece of paper ABCD is white on the front side and black on the back side and has an area of 3 square inches. Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is 1/2 white and 1/2 black. How far is A' from the fold line?

2. If 42% of all vehicles on the road last year were sports utility vehicles, and 73% of all single car rollover accidents involved sports utility vehicles, how much more likely was it for a sports utility vehicle to have such an accident than another vehicle?

Upon completion of the problems, the general questions were posed to further investigate their mathematical behaviors and the development of their problem solving abilities.

The open-ended questions contained a mix of both general questions that probed background and strategies (e.g., What actions, strategies, behaviors, beliefs most influence your success in solving problems?, What experiences have contributed most to the development of your problem solving abilities?)

and specific questions regarding the control mechanisms utilized during problem solving (e.g., How do you prioritize the actions you take during problem solving?, What techniques do you employ so that you are able to persist with a problem when you begin to feel frustrated?)

## Data Analysis and Results

The audio-tapes were transcribed and interview transcripts were analyzed. The analysis of the subjects' responses to the mathematical tasks involved the application of open coding strategies (Strauss & Corbin, 1998) to identify behaviors that emerged when responding to each mathematical task. Each of the identified behaviors was labeled using codes corresponding to each item in the theoretical framework. The collective behaviors of the 12 mathematicians were tabulated according to the frequency with which each behavior manifested itself during the problem solving process for each mathematician when completing each problem. The total occurrence of each behavior for each problem was totaled. The open-ended questions were coded using the Strauss and Corbin (1998) open and axial coding strategies.

The collective results for the 12 mathematicians when completing these four tasks revealed the highest occurrence of the following cognitive and metacognitive behaviors from the framework:

| | |
|---|---|
| CPE | Effort and energy is put forth to read and understand the problem. |
| CEM | Effort is put forth to stay mentally engaged |
| CEV | Voluntarily brings problem into consciousness and cognitive resources are expended |
| CES | Evidence of sense making/attempts to fit new information with existing schemata |
| CMQ | Reflects on the efficiency & effectiveness of cognitive activities |
| CMG | Generates conjectures |
| CMT | Tests conjectures |
| CMR | Relates problem to parallel problems and experiences |
| CMP | Refines, revises, or abandons plans as a result of what has happened in the solution process |
| CME | Manages emotional responses to the problem solving situation |

Additionally, these mathematicians were frequently observed exhibiting strong emotions, ranging from frustration, anxiety and impatience, to joy and pleasure. All participants were observed possessing a high level of mathematical integrity, as revealed by their care in offering only solution attempts with a logical foundation. Further, they were observed as possessing a high level of intimacy with particular problems. Once their curiosity was sparked, they expressed and were observed as being unwilling to let go of particular problems.

## Conclusions

These mathematicians demonstrated an initial and intense cognitive engagement when confronted with each of these problems. This was exhibited by their initial efforts to sort out the problem statements and organize the given information in the form of graphs, pictures and tables, as appropriate for the individual problem. Each mathematician invested varying amounts of time planning their solution approach, with the intensity and amount appearing to be influenced by their perception of the problem's complexity. Various metacognitive behaviors were exhibited routinely during the problem solving process. The subjects were observed evaluating the efficiency and effectiveness of their selected approach, while possessing a strong tendency to relate the given problem to other similar problems. During the problem solving process, each mathematician appeared to spontaneously employ self-reflection through a display of "internal discussions", and the weighing of the relative merits of various ideas. Conjectures were intermittently generated and systematically tested, although at times during their problem solving attempts, each mathematician exhibited mild frustration; However, their high confidence and use of coping mechanisms appeared to mitigate adverse affects. In fact, it was during these frustrating moments that they were most frequently observed scanning their knowledge base, making overt attempts to access potentially useful content.

Most notable among the mathematicians' problem solving behaviors was their unwillingness to let go of a problem once they had initiated a solution approach. This behavior was characterized by their frequently revisiting unsolved problem(s), and their continued pursuit of the solution, even after the interview was complete. Over half of the mathematicians sought out the interviewee the day following the interview to provide a written revised response to one of their unfinished problems. This exhibition of intimacy with particular problems also surfaced when analyzing the collection of responses to the general questions. When prompted to describe the factors that contributed most to their problem solving success, each mathematician described some form of regular and persistent cognitive engagement. As stated by one of the subjects, "..I just play with the problems and in that process things suggest themselves and connections get made in my mind...it's just that once a problem gets a hold of me it doesn't let go...I just keep it rattling around and sometimes something happens". The choice to repeatedly bring the problem into their consciousness was a crucial factor in the manifestation of exceptional problem solving power in this collection of mathematicians. Various factors influenced the emergence of a high level of problem intimacy; however pride and interest were observed as being strong influences for the majority of mathematicians in this study. As well, their high level of mathematical integrity, which manifested itself by offering only logically constructed solutions, appeared to promote focus and efficiency during the problem solving process

## References

Debellis, Valerie A. (1995). "Mathematical Intimacy: Local Affect in Powerful Problem Solvers".

Debellis, V.A., Goldin, G.A. (1998). "Aspects of affect: Mathematical intimacy, mathematical integrity", Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education.

DeFranco, Thomas C. (1996). "A Perspective on Mathematical Problem-Solving Expertise Based on the Performances of Male Ph.D. Mathematicians", *Research in Collegiate Mathematics Education II*, pp. 195-214.

Garofalo, J. and Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education*, 16, 163-176.

Geiger, V.; Galbraith, P. (1998). "Developing a Diagnostic Framework for Evaluation Student Approaches to Applied Mathematics Problems", International Journal of Mathematics Education, Science and Technology (29) 4, pp. 533-559.

Lester, F. K., Garafalo, J., and Kroll, D. L., (1989). *The role of metacognition in mathematical problem solving: A study of two grade seven classes*. Final report to the National Science Foundation of NSF project MDR 85-50346.

Lester, F.K. (1994). "Musings About Mathematical Problem-Solving Research: 1970-1994", *Journal for Research in Mathematics Education*, pp. 661-675.

Strauss & Corbin (1998). "Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory", Sage Publications.

Schoenfeld, Alan H. (1992). "Learning to Think Mathematically, Problem Solving, Metacognition and Sense Making in Mathematics". In D. Grouws, The handbook of research on the teaching and learning of mathematics, pp. 334-370.

Stillman, Gloria A. & Galbraith, Peter L. (1998). Applying mathematics with real world connections: Metacognitive characteristics of secondary students, *Educational Studies in Mathematics, 36*, 157-195.

Vinner, Shlomo (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Research Studies in Mathematics*, 34, 97-129.

# BRINGING OUT THE ALGEBRAIC CHARACTER OF ARITHMETIC: INSTANTIATING VARIABLES IN ADDITION AND SUBTRACTION[1]

David Carraher, TERC, Bárbara M. Brizuela, TERC & Harvard University, &
Analúcia D. Schliemann, Tufts University

*We report findings from a one-year teaching experiment designed to document and help nurture the early algebraic development of third grade students. We focus on an arithmetic problem that fixes some measures but allows more than one solution set. We highlight how children dealt with the fact that the quantitative relations referred to particular measures on one hand (and in that sense were arithmetical), and were meant to express general properties not bound to particular values, on the other (and in this sense were algebraic). We look at the role of instantiated variables in this tension and transition between the particular and the general.*

After carefully documenting the difficulties of algebra students (Booth, 1984; Da Rocha Falcão, 1992; Filloy & Rojano, 1989; Kieran, 1985a, 1989; Laborde, 1982; Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Sfard & Linchevsky, 1984; Steinberg, Sleeman & Ktorza, 1990; Vergnaud, Cortes, & Favre-Artigue, 1987), the field of mathematics education has gradually come to embrace the idea that algebra need not be postponed until adolescence (Bodanskii, 1991; Davis, 1985, 1989; Kaput, 1995; Vergnaud, 1988). Researchers have increasingly come to recognize that young children can understand mathematical concepts assumed to be fundamental to learning algebra (Brito Lima & da Rocha Falcão, 1997; Carraher, Schliemann, & Brizuela, 1999; Schifter, 1998; Schliemann, Carraher, Pendexter, & Brizuela, 1998). Many researchers and educators now believe that elementary algebraic ideas and notation may play an important role in students' understanding of early mathematics. To support this change in thinking and practice the field needs research on how young learners reason about problems of an algebraic or generalized nature.

The present research was undertaken as part of an early algebra study of the authors of this paper with a classroom of 18 third grade students at a public elementary school in the Boston area during a one-year teaching experiment. The school serves

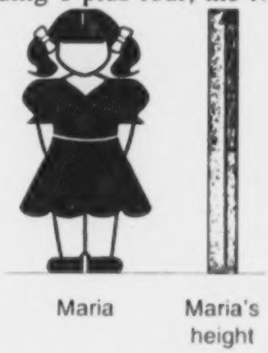
---

[1] Partial support for the research was provided by NSF Grant #9909591, awarded to D. Carraher and A. Schliemann, and NSF Grant #9722732, awarded to A. Schliemann. Support for a multimedia version of this paper was provided by NSF Grant 9805289, awarded to R. Nemirovsky and D. Carraher. Special thanks to Fred O'Meara, Thelma Davis, and the third grade students involved in our project for their help, and to Judah Schwartz (Harvard/MIT) and Pat Thompson (Vanderbilt) for constructive comments.

a diverse multiethnic and racial community reflected well in the class composition, which included children from South America, Asia, Europe, and North America. We had undertaken the work to understand and document issues of learning and teaching in an "algebrafied" (Kaput, 1995) arithmetical setting. Our activities in the classroom consisted of teaching a two-hour “math class” on a bi-weekly basis. The topics for the class sessions evolved from a combination of the curriculum content, the teacher's main goals for each semester, and the questions we brought to the table. We were well aware of the relative difficulty of additive comparison problems (Carpenter & Moser, 1982; Vergnaud, 1982) and seized upon additive comparisons as a special opportunity for approaching arithmetic from an algebraic standpoint.

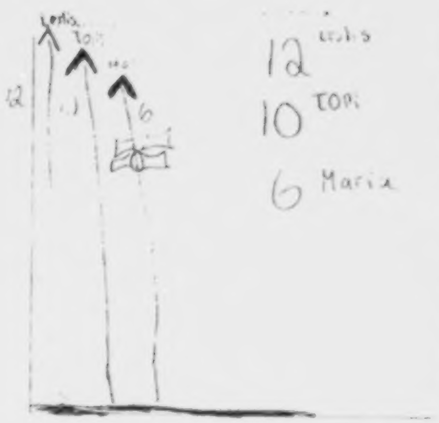
At the beginning of the fourth class, the problem below was shown to the children via projection onto a large screen. The children clearly understood that the problem entailed the heights of three children. However, many of the students were apparently treating the numbers as total heights. (“Tom is four inches tall…”). One child decided that Maria must be six inches tall and Leslie was 12 inches tall. Another volunteered that Tom was 10 inches tall, after adding 6 plus four, the two numbers given in the problem—although he was puzzled by how a child could be so short. The problem gave information about differences between heights; the children treated the difference information as actual heights. (This shift in level of discourse calls to mind what Thompson [1991] noted among fifth grade students, who mistakenly treated second-order differences as first-order differences.) As an attempt to get the children thinking about differences between heights, we decided to enact the problem in front of the class with three volunteers. In the course of this acting out, the three actors (students representing the people in the problem) helped the class gradually recognize the correct order of the children in the problem.

- ✓ Tom is 4 inches taller than Maria.
- ✓ Maria is 6 inches shorter than Leslie.
- ✓ Draw Tom's height, Maria's height, and Leslie's height.
- ✓ Show what the numbers 4 and 6 refer to.



When the pupils had established a consensus about the relative order of heights of the three protagonists, we asked them to provide us with notations showing the information from the problem, and to try to indicate where the numbers 4 and 6 should “go” in their diagrams.

When they were asked to represent the children in a drawing, 12 of the students assigned a value, in inches, to each member of the story. The diagram below typifies this. Note that 6 is taken to be Maria's height; but there is also a difference of 6 inches between Maria's and Leslie's heights. So this solution treats the numbers both as referring to actual heights as well as respecting the relative values.



Although only two numbers were given in the problem, every student who used numbers provided three, one each for Tom, Maria, and Leslie. (Actually, in two cases, children drew a series of numbers next to the children's height-arrows.) In 7 of the 12 cases where numbers were used, children provided heights for Leslie, Tom, and Maria that were consistent with the numbers given in the problem. But they did not label any parts of heights or intervals between heights as corresponding to these values. Even as they eventually came to realize that the measures, 4 in. and 6 in. referred to something other than total heights, many remained hesitant about what region of their diagram corresponded to these values[2].

In the following classroom example Kevin explains how he determined the heights.

Kevin: So she's (Leslie) taller than Tom by two inches. Mmhumm... cos Tom's taller than Maria by four inches... Mmhumm... and Maria is taller than, and Leslie is taller than Maria by six inches so that means that if he's taller than her by four inches and she, then Leslie has to be taller than Tom by two inches.

Bárbara: Mm... and to figure that out do you need to know exactly how high each one is?

Kevin: Yes

Bárbara: You do? How did you do it if you don't know how high they are?

Kevin: I do... cuz Maria's... Maria's...

Bárbara: Tell me exactly how high they are.

Kevin: Maria's... four feet six inches... and uh...

Bárbara: Where did you get that from?

Kevin: I don't know it just looks like that.

Bárbara: I think, I think you're just inventing that.

Kevin: I am.

Kevin instantiated Maria's height to 4 ft. 6 in. (was this another way to get the numbers 4 and 6 into the problem?) and then generated values for the other children consistent with the information about the relations among the heights.

---

[2] Adults understand that 4 inches corresponds to that part of Tom's height that surpasses Maria's height (ie. lies above her head). Even when the children understood that 4 inches was "how much bigger" Tom was than Maria, they tended to indicate where this value lay (4 in.) by pointing to the top of each head. They did likewise when showing the "difference" between their own classmates' heights. For many, a difference could not be shown as an interval along a spatial dimension; one had to re-enact the action of comparing!

Once the children distinguished between heights and their relative differences, we asked them to consider several scenarios or "stories". In "story 1" Tom's height was supposedly 34 inches; in story 2 his height was instead 37 inches; Maria and Leslie were each assigned heights of 37 inches in stories 3 and 4, respectively. In the final story Tom's height was listed simply as "T", which *we* took to refer to Tom's height *whatever* it might be. As far as we can tell, this was the first occasion in which the children were shown letters standing for unknown measures. The students quickly caught on to the idea that the first four stories were mutually exclusive: i.e., at most, one of them could be true. Within a few minutes they were able to determine heights consistent with the relative information provided at the beginning of the class. Our attention then turned to story 5, which, in *our* eyes, was a generalized description of each of the previous stories.

Kevin filled in the table, concluding that in each story Maria had 4 inches less than Tom. In the final case, where Tom's height was given as T, after some prodding from the interviewer, he noted that in each story Maria was 4 in. shorter than Tom, which he expressed as "T-4". Kevin then concluded that Leslie's height could be described by "T+2". His explanation shows him trying to find words for this new notation (note, for instance, the interpretation for T as "tall").

Kevin: Oh, OK, Tall. Tom's taller than Maria by 4 inches, and tall plus two equals... Leslie's taller than Tom by two inches (writing T+2 for Leslie).

Two other children correctly solved the general case but, when asked what T stood for, gave "tall" and "ten" as possible interpretations. We tried to explain that T could stand for "whatever Tom's height might be". This did not settle the issue, but over the next several classes we found students increasingly using the words "whatever" and "any" to explain the meaning of letters representing measures.

In the sixth and seventh weeks we interviewed children about issues related to the content of the different lessons. The interviews served as an additional source of data as well as an opportunity for the children to develop a greater understanding of problems similar to those presented in class. By then the students had become noticeably familiar with additive differences. Their understanding of unknowns and variables was still under development, as the following interview excerpt reveals. We enter the interview precisely when two pupils, Jennifer and Melissa, are explaining what "$x$+3" means. The context was a height problem in which Martha was said to be three inches taller than Alan.

David (interviewer): Why were we using $x$'s?

Jennifer: You might think like the $x$ is Alan's height and it could be *any* height.

This would seem to suggest that Jennifer has settled the basic idea of what a variable is. But shortly thereafter it becomes clear that there are still matters to consider.

David: OK, now, if I *didn't* know Alan's height, and I just had to say, "Well, I don't know it so I'll just call it '*x*'...."

Melissa: You could guess it.

Jennifer: You could say like, well it would just tell you to say *any* number.

David: Why don't I use an *x* and say whatever it is I'll just call it *x*? (umm)

Both: (puzzlement)

David: Do you like that idea, or does that feel strange?

Jennifer: It feels strange.

Melissa: No, I pretty much...[unlike her classmate, she is comfortable with using x in this manner.]

David (addressing Jennifer): It feels strange?

Jennifer: Yes, 'cause it has to, it has to have a number. 'Cause.... **Everybody in the world has a height.**

Jennifer seems to think that it would be inappropriate to use the letter *x*, representing *any* height, to describe Alan, since Alan cannot have *any* height; he must have a particular height. So the interviewer turns the discussion to another context to see if Jennifer's reaction extends beyond the particular case at hand.

David: Oh.... OK. Well,...I'll do it a little differently. I have a little bit of money in my pocket, OK—(to Jennifer) do you have any coins, like a nickel or something like that?

Jennifer: All mine's in the bag [in the classroom]...

David: OK, I'll tell you what: I'll take out, I'll take out a nickel here, OK. And I'll give that to you for now. I've got some money in here [in a wallet] can we call that x? (hmm) Because, *whatever it is*, it's that, it's the amount of money that I have.

Jennifer: **You can't call it x because it has... if it has some money in there, you can't just call it *x* because you have to count how many money [is] in there.**

David: But what if you don't know?

Jennifer: You open it and count it.

Jennifer insists that it would be improper to refer to the money in the wallet as *x*, since it holds a particular amount of money. (Melissa, who likens the value of x to "a surprise", experiences little, if any, conflict; *x* is simply a particular value that one does not know.) In a sense, Jennifer finds the example inconsistent with the idea of *x* as a variable, that is, something capable of taking on a range of values. And she has a point: there is something peculiar about the fact that a variable stands for many values, yet we exemplify or instantiate it by using an example for which only one value could hold (at a time). She eventually reduces her conflict by treating the amount of money in the wallet as, hypothetically, able to take on more than one value:

Jennifer: The amount of money in there is... *any* money in there. And after... if you like add five, if it was like... imagine if it was 50 cents, add five more and it would be 55 cents.

Fifty cents is only one of many amounts that the wallet *could have* (in principle) held.

## Concluding Remarks

Elementary algebra can be viewed as a generalization of an arithmetic of numbers and quantities. Among the types of generalization we would like to draw attention to are: (1) statements about sets of numbers (e.g. "all even numbers end in an even digit") as opposed to particular numbers (2) general statements about operations and functions (e.g. "the remainder of integer division is always smaller than the divisor") as opposed to particular computations; and (3) expressions that describe relations among variables and variable quantities (e.g. "y= x +3") as opposed to relations among particular magnitudes.

This paper describes an attempt to closely document how young learners reason about problems of an algebraic or generalized nature (of the third sort above).

In the heights problem, the letters T, M, and L represent the people in the problem, their particular heights, and any and all possible values their heights might take on. Although the problem was grounded in a story about particular actors, the real story is about the relations among them, regardless of their particular values. The whole point of using multiple "stories" or scenarios in the classroom example was to draw attention to the invariant properties of the problem as the heights of individuals took on different values (varied).

The children in the study had to deal with the fact that the quantitative relations referred to particular numbers and measures on one hand (and in that sense were arithmetical), and were meant to express general properties not bound to particular values, on the other (and in this sense were algebraic). Reasoning about variable *quantities* and their interrelations would seem to provide a stage on which the drama of mathematical variables and functions can be acted out. However, using worldly situations to model mathematical ideas and relations presents students with challenging issues. On some occasions children may be inclined (as Kevin, and Melissa initially were) to instantiate variables--to assign fixed values to what were *meant* to be variable quantities--without recognizing their general character. On others, they may find it strange (as Jennifer at one point did) to use particular instances to represent variables and functions when any instance is of a constant, unvarying nature.

In future research, we hope to explore appropriate scenarios for facilitating an understanding of variables and functions beyond particular instantiations. Through close descriptions such as those provided in this paper, we hope to help the mathematics education community of researchers and practitioners uncover the true potential behind an early introduction of algebraic concepts and notations.

**References:**

Bodanskii, F. (1991). The formation of an algebraic method of problem solving in primary school children. In V. Davidov (Ed.) Soviet studies in mathematics education, vol. 6: Psychological abilities of primary school children in learning mathematics. Reston, VA, NCTM, pp. 275-338.

Booth, L. (1984). Algebra: Children's Strategies and Errors. Windsor, UK, NFER-Nelson.

Brito-Lima, A. P. & da Rocha Falcão, J. T. (1997). Early development of algebraic representation among 6-13 year-old children: the importance of didactic contract. In Proceedings of the XXI International Conference Psychology of Mathematics Education, Lahti, Finland.

Carpenter, T. & Moser, J. (1983). The development of addition and subtraction problem solving skills. Em T. Carpenter, J. Moser & T. Romberg (Orgs.), *Addition and Subtraction: A Cognitive Perspective*. New York, Academic Press.

Carraher, D., Schliemann, A., & Brizuela, B. (1999). Bringing out the algebraic character of arithmetic. Paper presented at the 1999 AERA Meeting (40 pp.)

Da Rocha Falcao (1993). A algebra como ferramenta de representacao e resolucao de problemas. In A. Schliemann et al. (Eds.) Estudos em Psicologia da Educacao Matematica. Recife, Brazil, Editora Universitaria UFPE.

Davis, R. (1985). ICME-5 Report: Algebraic thinking in the early grades. Journal of Mathematical Behavior, 4, 195-208.

Davis, R. (1989). Theoretical considerations: Research studies in how humans think about algebra. In S. Wagner & C. Kieran (Eds.) Research Issues in the Learning and Teaching of Algebra, vol. 4. Reston, VA, NCTM/Erlbaum, pp. 266-274.

Filloy, E. e Rojano, T. (1989). Solving equations: the transition from arithmetic to algebra. For the Learning of Mathematics, 9(2), 19-25.

Kaput, J. & Blanton, M. (1999) Algebraic reasoning in the context of elementary mathematics: making it implementable on a massive scale." Paper given at the 1999 Annual Meeting of the AERA, Montreal, Canada.

Kaput, J. (1995). Transforming Algebra from an Engine of Inequity to an Engine of Mathematical Power By "Algebrafying" the K-12 Curriculum. Paper presented at the 1995 NCTM meeting, 17 pp.

Kieran, C. (1985). The equation-solving errors of novices and intermediate algebra students. Proceedings: IX International Conference Psychology of Mathematics Education. Montreal, Canada.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.) Research Issues in the Learning and Teaching of Algebra, vol. 4. Reston, VA, NCTM/Erlbaum, pp. 33-56.

Laborde, C. (1982). Langue naturelle et ecriture symbolique: deux codes en interaction dans l'enseignement mathematique. PhD Thesis, Universite Scientifique et Medicale, Grenoble, France.

Resnick, L., Cauzinille-Marmeche, E., & Mathieu, J. (1987). Understanding algebra. In J. Sloboda & D. Rogers (Eds.) Cognitive Processes in Mathematics. Oxford, Clarendon Press.

Schifter, D. (1998). Developing operation senses as a foundation for algebra I. Unpublished manuscript.

Schliemann, A.D., Carraher, D.W., Pendexter, W., & Brizuela, B. (1998) Solving Algebra Problems before Algebra Instruction. Paper presented at the Second Early Algebra Meeting, Tufts University/UMass-Dartmouth.

Sfard, A. & Linchevsky, L. (1984). The gains and the pitfalls of reification – The case of Algebra. Educational Studies in Mathematics. 26, 191-228.

Steinberg, R., Sleeman, D. & Ktorza, D. (1990). Algebra students knowledge of equivalence of equations. Journal for Research in Mathematics Education, 22(2), 112-121.

Thompson, P. (1991). Quantitative reasoning, complexity, and additive structures. Paper presented at the 1991 AERA meeting. Chicag, IL, April 3-7, 44 pp.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. Em T. Carpenter, J. Moser & T. Romberg (Orgs.) *Addition and Subtraction: A Cognitive Perspective*. New York, Academic Press.

Vergnaud, G. (1988). Long terme et court terme dans l'apprentissage de l'algebre. In C. Laborde (Ed.) Actes du premier colloque franco-allemand de didactique des mathematiques et de l'informatique. Paris, La Pense Sauvage, pp. 189-199.

Vergnaud, G., Cortes, A., & Favre-Artigue (1987). Introduction de l'algèbre auprès des débutants faibles: problèmes épistemologiques et didactiques. In G. Vergnaud, G. Brousseau, & M. Hulin (Eds.) Didactique et acquisitions des connaissances scientifiques: Actes du Colloque de Sèvres. Sèvres, France, La Pensé Sauvage, pp. 259-280.

# THE GAME OF SOCIAL INTERACTIONS IN STATISTICS LEARNING AND IN COGNITIVE DEVELOPMENT[1]

Carvalho, Carolina & César, Margarida
Centro de Investigação em Educação da Faculdade de Ciências da Universidade de Lisboa

**Abstract**: *This investigation is part of the project Interaction and Knowledge, whose main aim is to promote social interactions as one of the possible forms of pupils' socialisation, develop positive attitudes towards Mathematics, promote their socio-cognitive development and enhance their school achievement. Comparing the cognitive progress and mathematical performances of pupils working in dyads with those who did not work this way, the former show higher progress.*

## Introduction

In Portugal the current Mathematics curricula underline the need for pupils to develop mathematical skills in terms of knowledge, attitudes and abilities and find it important to create classroom practices where mathematical contents are apprehended and social and cognitive skills are developed (Abrantes, Serrazina and Oliveira, 1999). Social interactions, namely dyad work, have shown to be one of the possible forms of reaching this.

In the past two decades many investigations have discussed the role of social interactions in cognitive development and in promoting pupils' school achievement. A great deal of these studies began in a lab situation (César 1994, 1999), but the analysis of their results started to show heavy signs of their potential in the real-life, dynamic context of the classroom. This is how a strong movement arose among researchers of Social Genetic Psychology for assessing just how present the advantages revealed in the lab situation remained when the scenario was the school. However, there was still a long way to go, namely in understanding and explaining many of the mechanisms that are present in the social interactions taking place in the classroom, where the challenges are typical of lively, complex, dynamic contexts.

In Portugal, the studies launched by César (1994) and the attempt to respond to the challenges brought forth by teachers who wanted to know how to form groups for their pupils to reach a better performance led to a set of contextualised studies, which tried to assess the performance of different types of dyads, tasks and working instructions that subjects were given. At the same time, these studies allowed us to understand the mechanisms that are present in the classroom and showed the need to develop on-going work with the teachers at school. The project *Interaction and Knowledge* was born with this double aim: To continue the contextualised studies but with a semi-laboratorial character, for analysing in detail the mechanisms involved in interactions (Level 1); to create investigation dynamics with the teacher that allow for the use of some of the knowledge gained in Level 1 in their pedagogical practices (Level 2 which is an action-research project).

---

[1] The project Interaction and Knowledge was funded by IIE – Instituto de Inovação Educacional, in 1997 and 1998, and by CIEFCUL – Centro de Investigação em Educação da Faculdade de Ciências da Universidade de Lisboa, since 1996. We deeply thank all teachers and pupils who made this work possible.

## Theoretical background

The importance of social interactions has been studied in the last three decades. Doise, Mugny and Perret-Clermont (1975, 1976) and Perret-Clermont (1976/78) began the first studies showing the importance of peer interaction in the progress of cognitive development. The tasks used by then were still Piagetian ones and so they were not directly linked to *curricula* contents. Therefore their results could not explain much of what was going on when a pupil failed at school.

Vygotskian theory (1962, 1978), stressed the relevance of the context and social interactions in pupils' academic performances became more evident (Moll, 1990; Nunes and Bryant, 1997, Schneuwly and Bronckart, 1996). Vygotksy (1962, 1978) also introduced the concept of zone of proximal development (ZPD). He thought that teaching would be more effective if pupils worked in this zone, so one of the teacher's role was to identify the ZPD of each pupil and teach him/her according to it. One of the most exciting findings of the last decade was that peer interactions were even more powerful than what Vygotsky believed. Both symmetric and assymmetric dyads could lead to progression and both children of each peer could profit from the fact of interacting with each other (César, 1994, 1995, 1999).

In the last decades the importance of contextualized researches became clear and so school classes were a privileged stage for researchers. The tasks were no longer Piagetian, but directly related to *curricula* contents and psycho-social factors such as the situation, the task, the working instructions, the actors involved in the situation, the contents were submitted to an in-depth analysis. Performances were no longer seen as independent of these psycho-social factors and so social interactions played a significant role in the way they mediated pupils' relation with school knowledge.

Peer interactions were studied by many authors and they seemed to be quite effective promoting pupils' academic performances, namely in Mathematics (César, 1994, 1998a, 1998b, 1999; Perret-Clermont and Nicolet, 1988; Schubauer-Leoni and Perret-Clermont, 1997). Peer interactions were seen as a way of sharing and co-construction knowledge. They seemed to be a powerful way of confronting pupils with one another's solving strategies and so they had to make their conjectures and arguments. This need of giving explanations to his/her peer, the socio-cognitve conflict that could exist among children and the kind of interactive process they were engaged in may explain why working in dyads was so effective in many studies (Carvalho, 1998; César, 1998a, 1998b; César e Torres, 1997).

## Method

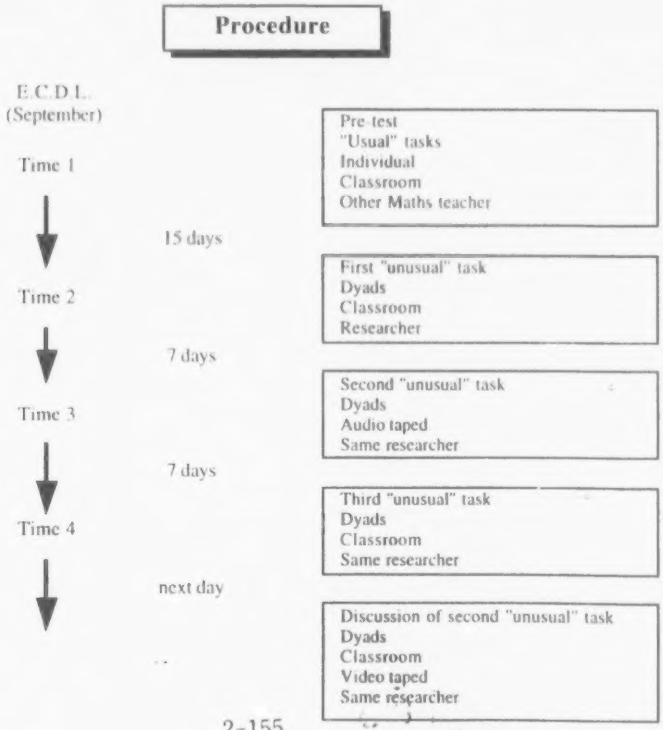
In this paper we will present results concerning Hypothesis 1 and 2 of this study, referring to the quantitative data. Hypothesis 1: verify if students of the experimental group (EG), who worked in dyads while solving statistical tasks, show more progress in their cognitive development than students in the control group (CG). Hypothesis 2: verify if students of the experimental group (EG), who worked in dyads while solving statistical tasks, show more progress between pre-test and post-test than students in the control group (CG).

### *Sample*

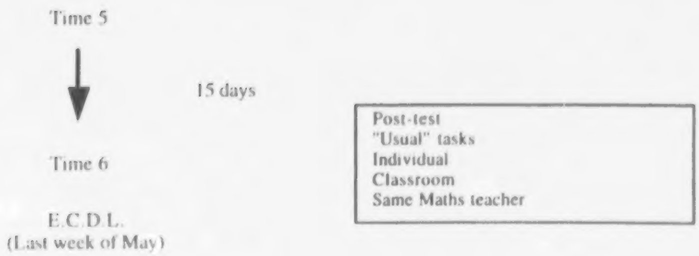
The sample was gathered during 1996/97 and 1997/98. .In the first year it was formed by 315 subjects (15 classes), in the second by 218 (10 classes), from two public schools on the outskirts of Lisbon. All subjects attended the compulsory 7th grade. Subjects' age ranged between 11 and 15 years (Average=12,5 and Sd=0,8).

*Instruments*
The Collective Test of Cognitive Development (E.C.D.L.) is a paper and pencil test created in France by J. Hornemann (1975), based on the Logical Development Scale (E.P.L.) built by F. Longeot (late 1960s). This developmental scale uses Piagetian-related tests. The E.C.D.L. evaluates logical thinking and allows us to determine the subject's level of cognitive development. It includes 4 sub-tests: crossings, lamps, drawings and letter game, each beckoning a certain type of operatory scheme: class intersection, propositions logic, co-ordination of a double system of co-ordinates and combination. The test classifies subjects according to five levels of development: less than concrete, concrete, intermediary, formal A and formal B. The test was adapted and standardized for the Portuguese population of Lisbon by César and Esgalhado (1986, 1987, 1988) and has been used in several Portuguese studies (César and Esgalhado, 1991; César, Camacho and Marcelino, 1993).
Statistical tasks were of two types: "usual", which teachers describe as typical exercises concerning this topic such as calculating the median, average, mode and interpreting and building tables and graphs (pre-test and post-test); "unusual", corresponding to more open and innovative tasks, more rarely used by teachers in the classroom (sessions of dyad work). Each of these tasks was classified in terms of three levels: weak, medium and high.

***Procedure***


Procedure
E.C.D.L.
(September)
Time 1
Pre-test
"Usual" tasks
Individual
Classroom
Other Maths teacher
15 days
Time 2
First "unusual" task
Dyads
Classroom
Researcher
7 days
Time 3
Second "unusual" task
Dyads
Audio taped
Same researcher
7 days
Time 4
Third "unusual" task
Dyads
Classroom
Same researcher
next day
Discussion of second "unusual" task
Dyads
Classroom
Video taped
Same researcher

BEST COPY AVAILABLE



## Results

In Table 1 we find cognitive development progress shown by subjects in the experimental group (EG) and by subjects of the control group (CG), in the 1996/1997 school year. Table 2 has the same kind of data regarding 1997/1998, when we replicated the study.

In both tables the results are distributed in two situations: progress and no progress. In the first situation we find subjects who maintained the same level of logical development, that is, who did not improve between the first and second application of the scale. In the second situation are subjects who went on to a higher level of logical development.

Table 1: Subject progress between the first and second applications of the E.C.D.L. (1996/1997)

| | No Progress | Progress | Total |
|---|---|---|---|
| CG | 91 (61%) | 59 (39%) | 150 (100%) |
| EG | 83 (50%) | 82 (50%) | 165 (100%) |
| µ | 174 | 141 | 315 |

Jonckheere test, Hypothesis: EG>CG, p=0.042

We may see that the experimental group (EG) subjects are those who show more progress when compared to control group subjects, the difference being significant (Jonckheere Test, p=0.042, in 1996/97; p=0,0000003 in 1997/1998).

Table 2: Subject progress between the first and second applications of the E.C.D.L. (1997/1998)

| | No Progress | Progress | Total |
|---|---|---|---|
| CG | 77 (70%) | 33 (30%) | 110 (100%) |
| EG | 38 (35%) | 70 (65%) | 108 (100%) |
| µ | 115 | 103 | 218 |

Jonckheere test, Hypothesis: EG>CG, p=0.0000003

So the experimental group (EG) subjects reveal greater progress in their cognitive development compared to the control group (CG) subjects, that is, students working in peer interaction present greater evolution.

Therefore, we may state that both in the year we carried out the first study (1996/97), as when we replicated it (1997/98), the subjects of the experimental group (EG) are those who show more cognitive development progress.

Tables 3 and 4 refer to the pre-test (before beginning the dyad work) and post-test (end of the experimental work) results of the subjects in the control group (CG) and the subjects in the experimental group (EG), in the two years during which the investigation took place. As we have already mentioned, the tasks used in pre- and post-test were usual exercises related to Statistics contents.

Just as in the case of cognitive development, the EG subjects show more progress between the pre-and post-test and regressed less between the two task applications. This means that when we assess the progress of subjects' mathematical performances, those working in dyads present a greater evolution.

Table 3: Subject progress between the pre-test and the post-test (1996/1997)

| | Regression | No Progress | Progress | Total |
|---|---|---|---|---|
| CG | 49 (15,5%) | 90 (28,5%) | 11 (3,5%) | 150 (47,6%) |
| EG | 8 (2,5%) | 79 (25%) | 78 (24,7%) | 165 (52,4%) |
| μ | 57 | 171 | 86 | 315 |

Jonckheere test, Hypothesis: EG>CG, p=0.0000000

Table 4: Subject progress between the pre-test and the post-test (1997/1998)

| | Regression | No Progress | Progress | Total |
|---|---|---|---|---|
| CG | 37 (17%) | 56 (25,7%) | 17 (7,8%) | 110 (50,4%) |
| EG | 2 (0,9%) | 39 (17,9%) | 67 (30,7%) | 108 (49,5%) |
| μ | 39 | 95 | 84 | 218 |

Jonckheere test, Hypothesis: EG>CG, p=0.0000317

But it is best to keep in mind that in both years and in both groups there were pupils who were already in the high level, so they could not move onto an even higher level, that is, they could not show progress in terms of achievement. Therefore the fact that the difference between the two groups was statistically significant is even more relevant.

## Discussion

The data presented here confirm Hypothesis 1 and 2: subjects in the experimental group (EG) show more progress regarding cognitive development and mathematical performance than those in the control group (CG). As the experimental group subjects were those who worked in dyads while solving intermediary tasks, we may consider, as Iannaccone and Perret-Clermont (1993) declare, that the results of our investigation suggest that social interactions facilitate the promotion of cognitive development and mathematical performance.

In order to try to explain this progress we are led to reflect upon the mechanisms at stake in social interactions. The concept of socio-cognitive conflict, introduced by Doise, Mugny and Perret-Clermont (1975, 1976) allow us to understand the dynamics between the social and the individual which are present when one subject is confronted with another, regarding a task they must solve together, with the knowledge and skills each one has, all this happening in a social context that isn't neutral as there are where positions of power and leadership that are also established.

The conflictual side of interaction occurs on two levels: a cognitive one, allowing for cognitive restructuring resulting from the interaction, influencing the subject's operatory development; a social one, caused by the fact that subjects have to be capable of managing the interaction itself, that is, disagreement, consensus or leadership. Thus, there is socio-cognitive progress and improvement in subjects' Maths performances (César, 1998a; 1998b, 1999; César and Torres, 1997, 1998; Perret-Clermont, 1976/78) which remain even when subjects go back to working individually or on tasks of a different nature from that which was established during interaction.

When students interact among each other, as happens with those in the experimental group, they have more opportunities to confront each other about their personal points of view regarding different ways of solving a task, negotiating a meaning and managing an interpersonal relationship. From this process arises a double unbalance: on the one hand, interpersonal, that is, between two individual's answers; on the other hand, intra-individual, when one subject is invited to question his/her answer when faced with another possible answer, found by his/her partner. This double mechanism of unbalance is much more than a mere opposition between two social partners faced with performing a task.

It is precisely while co-ordinating different points of view or, as Gilly (1988) puts it, in the attempt to overcome the inter-individual cognitive unbalance, that subjects manage to solve their own intra-individual cognitive conflict. Quoting Doise and Mugny (1981), it is through *the interiorization of social co-ordinations that one reaches new intra-individual co-ordinations* (p.23).

In earlier studies (César, 1994, 1995, 1999), we found that there are types of dyads, working instructions and tasks which are more favourable in terms of subject progress. Those showing more progress are dyads with interaction, encouraged to discuss their points of view from the beginning, debating solving strategies until some agreement is reached. On the other hand, more open tasks, which we call "unusual", also facilitate students' progress. Another fact that seems to cause pupils' progress is the general discussion that occurs after pupils have solved the task two by two, in dyads. Having to explain and defend their positions to other dyads and to the teacher is a fundamental moment of communication and construction of a collective intersubjectivity that facilitates interiorization and the apprehension of information and knowledge brought about by the discussion. This general discussion opens the possibilities of socio-cognitive (inter-individual) and facilitates cognitive (intra-individual) conflicts, that is, it broadens the mechanisms that are present in social interactions and that apply when working in dyads. This explains why the results we had are even more pronounced than in previous studies (César, 1994), for in this study we only used the types of task, dyad and working instructions that had shown to be more promising. Besides this, we added a general final discussion, which also contributed towards better results (César, 1999; Johnson and Johnson, 1984, 1989; Slavin, 1980, 1990).

Thus, subjects' progress in terms of cognitive development and mathematical performance seems to be due to two sets of factors: on the one hand, those held by the developmental theory which explain that even those belonging to the control group (CG) may reach a higher level in the second application at the end of the year; on the other hand, the interactions established between peers, working instructions and the type of proposed tasks, which promote the establishment of a socio-cognitive conflict, making subjects seek a common intersubjectivity which facilitates their cognitive and mathematical progress. However, since the difference in progress found between subjects of the experimental group (EG) and those of the control group (CG) is very significant, both in the first year of the study and during its replication, we may say that peer interaction is an efficient way of promoting subjects' cognitive development and mathematical performance.

Finally, we thought we should stress a particularly important aspect from the psychological and pedagogical point of view: for cognitive progress to take place, social interaction doesn't have to be established with a more competent peer, as Vygotsky claimed (1962, 1978). Even in symmetric dyads there is cognitive progress and there are more competent peers who also show progress. Therefore, what seems essential for cognitive progress to take place is the quality of the interaction, the ability subjects have to work in their zone of proximal development, to find and discuss meanings, solving strategies, mathematical knowledge and how they are capable, or not, from a social point of view, of managing the interaction.

## Final Remarks

Results presented in this study show how peer interaction can promote subjects' cognitive development and mathematical performance (Carvalho e César, 1999a; 1999b). When subjects have to face their conjectures and their respective arguments they are forced to decenter from their personal positions in order to understand other ways of solving tasks and to explain their personal solutions. This way, the socio-cognitive conflict which is present when working in dyad in the classroom context and faced with "unusual" tasks promotes subjects' operatory development, as well as their mathematical performance.

The importance of social interactions in students' development of cognitive structures is undeniable. Although *opposite solutions* which arise when students work in interaction on a task *aren't the only way of producing intra-psychic unbalances* (Gilly, 1988, p.23), letting students work interactively in the classroom is one of the teaching practices with most potential for promoting subjects' development and their mathematical knowledge.

However, in most Portuguese schools, despite students being sat in pairs, we usually find a great deal of vertical interactions, where the teacher interacting with students always keeps a position of leadership regarding the interactive process. Students aren't used to interacting among each other, adopting a horizontal structure, that is, accepting a reciprocity in peers' social statutes. So when they try this form of solving mathematical tasks they show considerable progress in their cognitive development and a mathematical performance. Therefore, implementing this sort of work is a possible solution for promoting students' achievement.

### References

Abrantes, P., Serrazina, L., & Oliveira, I., (1999). *A matemática na educação básica.* Lisboa: Ministério da Educação, Departamento de Educação Básica.

Carvalho, C. & César, M. (1999a). Reflexões em torno de dinâmicas de interacção: O caso do trabalho em díade em tarefas não-habituais de Estatística. *Proceedings from VIII Encontro de Investigação em Educação Matemática* (In press).

Carvalho, C. & César, M. (1999b). *Peer Interaction, Mathematical and Cognitive Development.* Poster presented in E.A.R. L.I,., Goteborg, Suécia, 24-28 Agost 1999.

Carvalho, C. & César, M. (1999b) Interacções Socias: Que mitos? Que realidades?.*Proceedings from Profmat 99* (pp.329-334) Portimão: APM.

César, M. (1994). *O papel da interacção entre pares na resolução de tarefas matemáticas*. Lisboa: FCUL (Phd Thesis).

César, M. (1995). Interacção entre pares e resolução de tarefas matemáticas, *Proceedings from VI Seminário de Investigação em Educação Matemática* (pp.225-240). Lisboa: APM.

César, M. (1998a). Y se aprendo contigo? Interacciones entre parejas en el aula de matemáticas. *Uno* 16, 11-23.

César, M. (1998b). Social Interactions and Mathematics Learning. *MEAS 1 Proceedings* (pp.110-119). Nottingham: Nottingham University.

César, M. (1999). Interacções sociais e apreensão de conhecimentos matemáticos: a investigação contextualizada. *Proceedings from* Escola de Verão de Educação Matemática. Lisboa: SPCE/SEM (in press).

César, M., Camacho, H. & Marcelino, T. (1993). Desenvolvimento cognitivo e sucesso escolar num meio socio-culturalmente desfavorecido. *Aprendizagem e Desenvolvimento, III*(13/14), 149-152.

César, M. & Esgalhado, A. (1986). Symbolisme et développement cognitif- quelques donnés a propos de la traduction et de l'adaptation l'ECDL. Poster presented in the 8ème Cours Avancé de la Fondation des Archives Jean Piaget, Genève, Suiça, 22-26 de Setembro de 1986.

César, M. & Esgalhado, A. (1987). *L'ECDL et les différences Individuelles*. Poster presented in the 9ème Cours Avancé de la Fondation des Archives Jean Piaget, Genève, Suiça, 21-25 de Setembro de 1987.

César, M. & Esgalhado, A. (1988). Desenvolvimento cognitivo e insucesso escolar. *Proceedings from Encontro Internacional de Intervenção Psicológica na Educação* (pp.365-371*)*. Porto: APPORT.

César, M. & Esgalhado, A. (1991). Desenvolvimento Cognitivo e Percurso Escolar. *Revista de Educação, II*(1), 57-61.

César, M. & Torres, M. (1997). Students interactions in math class. *Proceedings from CIEAEM 49* (pp.76-85*)*. Setúbal: ESE de Setúbal.

César, M. & Torres, M. (1998). Actividades em interacção na sala de aula de matemática. *Proceedings from VI Encontro Nacional de Investigação em Educação Matemática* (pp.71- 87). Portalegre: SPCE.

Doise, W., Mugny, G. & Perret-Clermont, A.- N. (1975). Social interaction and the development of cognitive operations. *European Journal of Social Psychology*, *5*(3), 367-383.

Doise, W., Mugny, G. & Perret-Clermont, A.- N. (1976). Social interaction and cognitive development: further evidence. *European Journal of Social Psychology*, *6*(2), 245-247.

Doise, W. & Mugny, G. (1981). *Le Développment Social de l'Intelligence*. Paris: Interéditions.

Gilly, M. (1988). Interactions entre pairs et constructions cognitives: modeles explicatifs. In: A-N. Perret-Clermont, & M. Nicolet, (1988). *Interagir et Connaitre* (pp.19-28). Fribourg: Del Val.

Iannaccone,A. & Perret-Clermont, A.-N. (1993).Qu'est-ce qui s'apprend? Qu'est-ce qui se développe? In: J.Wassmann, P.R. Dasen (1993). *Les Savoirs quotidiens, les approches cognitives dans le dialogue interdisciplinaire* (pp.235-257). Freiburg: Schweiz.

Johnson, D. W. & Johnson, R.T. (1984). Building acceptance of differences between Handicapped and nonhandicapped students: the effects of cooperative and individualistic instrution. *The Journal of Social Psychology*, *7*(122), 257-267.

Johnson, D. W. & Johnson, R.T. (1989). *Cooperation and competition: a meta-analysis of the research*. Hillsdale, New York: Lawrence Erlbaum.

Moll, L.C. (1990). *Vygotsky and Education*. Cambridge UK: Cambridge University Press.

Nunes, T. & Bryant, P.,(1997). *Learning and Teaching Mathematics: An international perspective*. Hove: Psychology Press.

Perret-Clermont, A.-N. (1976/1978). *Desenvolvimento cognitivo da inteligência e interacção social*. Lisboa: Instituto Piaget.

Perret-Clermont, A.-N. (1988). *Interagir et Connaitre*. Fribourg: Del Val.

Schneuwly, B. & Bronckart, J.-P. (1996). *Vygotsky aujourd'hui*. Neuchâtel: Delachaux et Niestlé.

Schubauer-Leoni, M.L. & Perret-Clermont, A.-N. (1997). Social interactions and mathemathical learning. In T. Nunes & P. Bryant (Eds.), *Learning and Teaching Mathematics: An international perspective* (pp. 265-283). Hove: Psychology Press.

Slavin, R.E. (1980). Cooperative learning. *Review of Educational Research, 50* (2), 315-342.

Slavin, R.E. (1990). Research on cooperative learning: consensus and controversy. *Educational Leadership*, 47, 52-54.

Vygotsky, L. (1962). *Thought and Language*. Cambridge MA: MIT Press.

Vygotsky, L. (1978). *Mind and Society*. Cambridge MA: Harvard University Press.

Werstch (1991). *Voices of Mind*. Cambridge MA: Harvard University Press.

BEST COPY AVAILABLE

# STUDENT MISCONCEPTIONS IN INTERPRETING BASIC GRAPHIC CALCULATOR DISPLAYS[1]

Michael Cavanagh & Michael Mitchelmore
*Macquarie University, Australia*

*We investigated how students interpret linear and quadratic graphs on a graphics calculator screen. Clinical interviews were conducted with 25 Grade 10-11 students as they carried out a number of graphical tasks. Three common misconceptions were identified: a tendency to accept the graphic image uncritically, without attempting to relate it to other symbolic or numerical information; a poor understanding of the concept of scale; and an inadequate grasp of accuracy and approximation.*

Graphics calculators (GCs) were first developed in the mid 1980s and since that time have become steadily cheaper, more user-friendly, and more powerful. As a result, they are being increasingly used in mathematics teaching. While there is no shortage of research on the use of GCs in the classroom (Dunham & Dick, 1994; Penglase & Arnold, 1996), many of these studies compare the use of a GC to that of pen and paper on the same tasks. We have found no research (not even in PME Proceedings) which investigate how students use GCs and the mathematical misconceptions which may arise. The research reported below is a first attempt to rectify this shortcoming. We focus on the use of GCs for graphing functions prior to calculus.

## Background

Graphs are commonly regarded as a crucial weapon in the mathematician's arsenal. Research on student understanding of graphs (Leinhardt, Zaslavsky, & Stein, 1990) has led to the conclusion that their major problem is learning the links between graphs and other representations of functions (Dugdale, 1993; Moschkovich, Schoenfeld, & Arcavi, 1993). These links are continually called into play, from reading values off a graph to predicting the shape of a graph from its equation.

In precalculus mathematics, much attention is paid to graphs of linear and quadratic functions. Students learn to draw such graphs fluently and link the symbolic form to their shape (line, parabola, or inverted parabola) and details (axis intercepts, slope of a line, vertex of a parabola). When GCs are not available, examples have to be carefully chosen to minimise distractions. Thus, scales are always "nice" (e.g., 1 unit to 20 mm) and usually the same on both axes. The functions are chosen so that the interesting parts of their graphs occur fairly near the origin, and the coordinates of critical features (e.g., intersections with the axes) are almost always small integers. With GCs, students have to cope with "nasty" and unsymmetrical scales, blank

---

[1] This project is supported by a Strategic Partnerships with Industry—Research and Training (SPIRT) grant from the Australian Research Council. The industry partner is Shriro Australia Pty. Ltd. (distributors of Casio calculators).

BEST COPY AVAILABLE

screens or partial views, and non-integer coordinates. Because students are not plotting individual points, with the absence of labeled axes or even grid lines, the whole process might appear magical (Dion & Fetta, 1993). There is clearly a danger that fundamental misconceptions might arise.

A small number of studies have noted student misconceptions. Tuska (1993) analysed the responses of first year college students on multiple choice examinations to determine which errors might be related to GC use. She identified misconceptions in students' understanding of the domain of a function, asymptotes, the solution of inequalities, and the belief that every number is rational. Williams (1993) also identified similar difficulties. Steele (1994) noted the ready acceptance by students of the initial graph shown in the default window of the GC. Vonder Embse and Engebretsen (1996, p. 508) describe a situation where a pair of perpendicular linear graphs do not appear at right-angles on the GC screen because of the unequal scales on the axes. Goldenberg and Kliman (1988) also identify misunderstanding of scale as a source of misconceptions. The present study was designed to investigate such misconceptions in detail.

## Method

*Sample.* Clinical interviews were conducted with 5 students from each of 5 metropolitan high schools in Sydney (15 students were in Year 10: 8 girls and 7 boys, and 10 students were in Year 11: 5 girls and 5 boys). The students were all studying mathematics at the highest level available to them and were drawn from the higher ability classes in each participating school. (It was felt that these students would be better able to respond to the challenge of the interview tasks and articulate their ideas more fully.) It was assumed that the misconceptions which these students demonstrated would also be found in students of lower ability.

All the students had studied the graphs of straight lines and the gradient-intercept equation $y = mx + b$. They had sketched the graphs of parabolas given in the general form $y = ax^2 + bx + c$ and were familiar with the quadratic formula.

The students had used the Casio *fx*-7400G graphics calculator in their mathematics lessons for between 6 and 12 months prior to the first interview. In one school, the students owned their own *fx*-7400G and could use it in all lessons and examinations; in all the other schools the students had only limited access to a class set of GCs. Broadly speaking, the students were inexperienced users of the technology; they had only used them to display graphs of linear and quadratic functions.

*The interviews.* Dunham and Dick (1994) suggest that "by probing students' conceptual understanding through interviews, researchers can paint a more detailed picture of the effects of graphing calculator-base instruction on students' learning" (p.441). The interviews in this study were designed to confront some of the limitations of the GC directly and create a problematic situation for students. Each student was interviewed individually by the first author for fifty minutes on three separate occasions, each approximately two weeks apart. The students completed a variety of graphing tasks and were asked to explain the processes they used to interpret the graphs they saw. All the interviews were videotaped.

In this paper, we report on students' responses to the tasks dealing with (a) partial views, (b) non-symmetrical scales, and (c) non-integer features. The relevant tasks were as follows:

1. Draw a sketch of $y = 0.1x^2 + 2x - 4$. You may use the graphics calculator to help you.

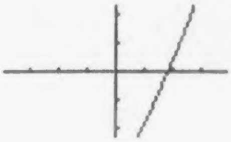


*Figure 1*

The image of this function produced by the GC in the initial window is shown in Figure 1. Task 1 tested whether or not the students could refer to the algebraic representation of the function and recognise that the GC screen did not display a complete graph of the parabola. This task also examined what students might do to obtain a more representative graph.

2. Use the graphics calculator to find the intersection of $y = 2x - 1.5$ and $y = 3x + 0.8$.

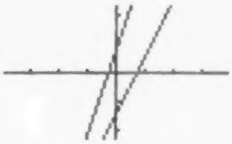


*Figure 2*

The lines are shown in the initial window in Figure 2. Task 2 was designed to see how the students would respond to the visual cue of the lines coming together at the bottom of the screen and if they would move beyond the initial window to locate their intersection. This task also investigated how students might interpret two lines which appear to meld together over a number of pixels.

3. Display the graph of $y = 0.75x^2 - 1.455x - 1$ on the graphics calculator. Find the intercept with the positive $x$-axis and the coordinates of the vertex of this parabola.

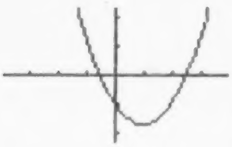


*Figure 3*

Figure 3 shows the parabola in the initial window. The quadratic function was chosen with irrational roots, to see how the students would interpret the decimal coordinates displayed on the GC for the $x$-intercept. The graph also appeared with a flat line of six pixels near the vertex rather than a single lowest pixel, to investigate how the students would deal with that situation.

4. Explain why the graphs of $y = 2x + 3$ and $y = -0.5x - 2.5$ do not appear at right angles on the screen. What could you do to make the lines look more perpendicular?

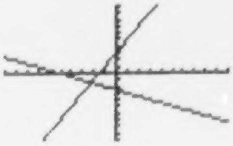


Figure 4

Figure 4 shows the screen for Task 4, with the $x$- and $y$-axes displayed from -10 to 10. Task 4 was designed to investigate how the students would deal with graphs displayed on axes which are not scaled symmetrically.

5. Which, if any, of the calculator screens shows the line $y = x$?

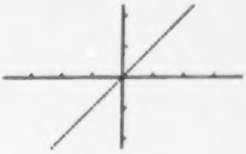


Figure 5(a)

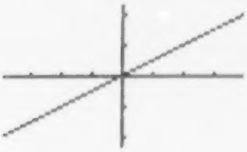


Figure 5(b)

Figure 5 shows the two windows which the students were given in Task 5. Figure 5(a) shows the line $y = x$ in the initial window, and Figure 5(b) shows the same line in a window where both the scale and the interval between the tick marks on the y-axis have been doubled.

The purpose of this task was to investigate any assumptions which the students held about the format of the window of the GC. Task 5 also provided a further opportunity to discuss the students' notions of scale.

6. Display the graph of $y = x^2 - 2x + 3$ on the graphics calculator. Can you change the window settings of the calculator so that the graph appears as a horizontal line?

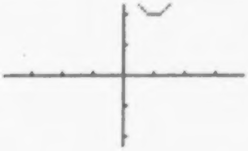


Figure 6

Figure 6 shows the graph of the parabola as it appeared in the initial window. The quadratic function was chosen to appear entirely within one quadrant in the initial window so that students would need to move the origin away from the centre of the screen.

## Results

*Task 1.* Seven students (28%) sketched a parabola; all commented that the $x^2$ term in the function indicated that the graph must be a parabola and then zoomed out until they saw the U-shaped curve. The remaining 18 students (72%) drew a straight line as their sketch of the quadratic function. Of these, 5 students (20%) simply copied the straight line directly from the calculator screen; the other 13 students either zoomed out ($n = 6$) or referred to the constant term in the function ($n = 7$) to find the $y$-intercept and marked it on their sketch.

*Task 2.* All 25 students (100%) moved beyond the initial window in Figure 2, either by scrolling the window down or zooming out to search for the intersection point. Students then had to interpret the group of pixels where the two lines overlapped. Five students (20%) thought that the lines had more than one point of intersection. The remaining students attempted to choose a point near the middle; 5 (20%) found the correct solution (-2.3, -6.1) and toggled between the lines to check their result, but 13 (52%) simply chose a point with "nice" coordinates.

*Task 3.* 19 students (76%) averaged the $x$-coordinates of the pixels immediately above and below the $x$-axis to obtain an estimate for the intercept, despite the fact that the $y$-coordinate displayed for one pixel was considerably closer to zero than the other. In a similar way, 21 students (84%) explained that the $x$-coordinate at the vertex of the parabola could be found by averaging the $x$-values of the two centre pixels in the row at the base of the graph in Figure 3. The students did not seem to realise that the $y$-coordinate of this point was greater than the $y$-coordinate of the lower of the two centre pixels.

All the students then zoomed in to find the intercept and vertex more accurately. Most (88%) continued zooming until the interviewer stopped them. They all expressed the belief that the relevant point could be found exactly provided one was prepared to zoom in a sufficient number of times. Only three students (12%) correctly stated that the roots of the parabola were irrational and that the $x$-intercept was therefore a non-terminating decimal which could not be found exactly. Only one of these students could also explain that the $x$-coordinate of the vertex of the parabola was a rational number which could be expressed by a terminating decimal.

*Task 4.* Only 4 students (16%) explained that the angle between the lines shown in Figure 4 was due to the unequal scaling of the coordinate axes without being prompted. The remaining 21 students (84%) eventually recognised that the axes were not scaled equally and that this explained why the lines did not appear perpendicular.

*Task 5.* 92% of the students ($n = 23$) thought that the screen shown in Figure 5(a) was the line $y = x$ and that Figure 5(b) must represent the line $y = 0.5x$. Each of these students explained that the tick marks calibrated on the coordinate axes must always represent unit values. Only 2 students (8%) recognised that the graphs were an artifact of the scale of the axes.

*Task 6.* This task was only attempted by 12 students. Only one student was able change the window settings so that the parabola appeared as a horizontal line across the screen. All of the remaining 11 students were unable to complete the task because they wanted to maintain the origin at the centre of the screen: Each time they reset the window, they chose a negative value for the minimum and the corresponding positive value for the maximum.

## Discussion

Student responses to the tasks reported above suggest three areas of conceptual difficulty in using GCs.

*The strength of the visual image.* Throughout the interviews, students tended to accept whatever was displayed in the initial window without question and did not relate the graphs they saw to the algebraic representation of the function. Sometimes this was misleading, as in Task 1, and sometimes not, as in Task 2. The difference in the results of Task 1 and Task 2 may be due to the fact that Task 2 incorporates a visual cue which does not require any reference to the symbolic representation of the functions in order to recognise that one needs to display more of the graph.

The power of the GC screen was also evident in Tasks 4-6. We note that the difficulties which students experienced on these tasks simply would not have occurred in the traditional introduction to linear and quadratic graphs without GCs.

In general, students appeared to defer to the GC in preference to their general knowledge of functions. They were more likely to zoom in on an important feature of a graph than to zoom out to ensure that a complete graph of a given function was displayed.

*Scale.* The students seemed to have little or no experience in dealing with graphs which were drawn on axes which were not scaled equally (see Leinhardt, Zaslavsky & Stein, 1990). Whenever they were required to draw a graph by hand during the interviews, they always did so using symmetrically scaled axes which were invariably marked at unit intervals.

An important feature of the students' ability to comprehend the concept of scale is the distinction between what might be called a *relative* or *absolute* understanding of scale. The former correctly regards scale as a ratio of distance to value, while the latter interprets scale solely as either the measure of the distance between adjacent markings on the axes or their value.

The vast majority of students had only developed an absolute understanding of scale. They found it difficult to give appropriate or meaningful descriptions of the scale of the coordinate axes displayed in the window, and they explained their ideas about scale almost exclusively in the language normally associated with simple scale drawings. These drawings are essentially just enlargements or reductions where the symmetry of the original object is always maintained and a single value, the enlargement factor, is a completely adequate description of the scale change. This is too limited a context for developing the skills needed to progress to a relative concept image of scale; it does not sufficiently equip students to deal with the asymmetric graphs which inevitably arise within the rectangular window of the GC.

*Approximations.* Dick (1992) argues that students will need to acquire new kinds of numerical skills if they are to use GCs effectively as problem solving tools. The results of this study certainly support that view. The students who were interviewed often had great difficulty in making appropriate numerical estimates for the values they were looking for. The most common approach used by the students was to average the coordinate values on either side of a point of interest regardless of whether the actual value was closer to one side or the other.

Students also regularly made a direct correlation between the greater number of significant figures given in a decimal value and its accuracy, without any real attempt to consider the specific context of the problem they were being asked to solve. Somewhat paradoxically, however, the students showed a marked preference for integer or other "nice" values when locating intersection points or intercepts.

In general, the students thought that all points on a graph must always be expressed by finite decimal values because they represented exact distances from the origin on the number plane. Few students were aware of the differences in the decimal representations of rational and irrational numbers.

## Implications and Conclusions

The results of this study indicate that there may be a need for a stronger curricular emphasis on scale. Students need more experience in controlling scale and watching for its effects on graphs. A greater use of asymmetric scales should also be encouraged. Number theory, particularly relating irrational numbers to their decimal approximations, may need more attention in the syllabus as well.

It is often claimed that the visual images provided by the GC can strengthen students' conceptual understanding. However, visual images can also be misleading and students who are not strong visualisers could find such images problematic. A more detailed study of the student interviews may give further insights into when GC images are helpful and when not.

Given the misconceptions we have identified, the question arises: Should mathematics teachers attempt to structure the examples students work so that they avoid such difficulties? Or should they deliberately plan activities which force students to face them? We believe that teachers should encourage students to confront the limitations of the technology: Doing so would not only strengthen students' basic understanding but could lead to so much interesting mathematics. But clearly we need to take care in how we challenge students' misconceptions.

Misconceptions may also lessen with greater exposure to GCs. The findings of Ruthven (1990) that regular access to the technology can have a positive influence on linking different representations of functions were generally supported by the results of this research project: The students who owned their own GCs more frequently exhibited a critical awareness of the calculator's output.

Finally, one of the things which struck us most during the interviews is that we should never assume too much about what our students perceive when they look at the screen. What may seem obvious to the mathematically experienced may not be equally apparent to novice learners. For instance, more than one student struggled to explain how the coordinate axes in Figure 4 could both range from -10 to 10 and yet look so different; they had simply failed to notice the rectangular shape of the window.

## References

Dick, T. (1992). Super calculators: Implications for the calculus curriculum, instruction, and assessment. In J. T. Fey & C. R. Hirsch (Eds.), *Calculators in mathematics education: 1992 yearbook* (pp. 145-157). Reston, VA: National Council of Teachers of Mathematics.

Dion, G. S., & Fetta, I. B. (1993). Calculator graphs: Magic? or mathematics? *Mathematics and Computer Education, 27(3)*, 180-197.

Dugdale, S. (1993). Functions and graphs - perspectives on student thinking. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp.101-130). Hillsdale, NJ: Lawrence Erlbaum Associates.

Dunham, P., & Dick, T. (1994). Research on graphing calculators. *The Mathematics Teacher, 87(6)*, 440-445.

Goldenberg, E. P., & Kliman, M. (1988). Metaphors for understanding graphs: What you see is what you see. (Report No. ETC-TR-88-22). Cambridge, MA: Harvard Graduate School of Education, Educational Technology Center. (ERIC Document Reproduction Service No. ED 303 369).

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60(1)*, 1-64.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp.69-100). Hillsdale, NJ: Lawrence Erlbaum Associates.

Penglase, M., & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. *Mathematics Education Research Journal, 8(1)*, 58-91.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics, 21*, 431-450.

Steele, D. (1994). *The Wesley College technology enriched graphing project.* (Unpublished Master's thesis, University of Melbourne, 1994).

Tuska, A. (1993). *Students' errors in graphing calculator-based precalculus classes.* (Doctoral dissertation, The Ohio State University, 1992). Dissertation Abstracts International, 53/08, 2725.

Vonder Embse, C., & Engebretsen, A. (1996). Friendly windows for graphing calculators. *The Mathematics Teacher, 89(6)*, 508-511.

Williams, C. G. (1993). Looking over their shoulders: Some difficulties students have with graphing calculators. *Mathematics and Computer Education, 27(3)*, 198-202.

BEST COPY AVAILABLE

# STEP SKIPPING DURING THE SOLUTION OF PARTITIVE QUOTIENT FRACTION PROBLEMS

Kathy Charles Rod Nason
Queensland University of Technology Queensland University of Technology

*Abstract: In this paper, we report on a study which investigated the step skipping manifested by two grade three students engaged in solving partitive quotient fraction problems. Using a SOAR production system framework, we found that some step skipping facilitated the construction of the partitive quotient fraction construct while others inhibited this process of abstraction. A number of implications for future research and teaching are identified and discussed in the concluding section of this paper.*

## INTRODUCTION

As learners solve the same types of problem again and again, not only do they get faster at doing the problems, but very often the process they use to solve the problems changes as well, often resulting in skipped steps (Koedinger & Anderson, 1990). This reorganisation and skipping of steps helps learners to solve problems more quickly, efficiently and easily (Blessing & Anderson, 1996).

Step skipping is often thought of as a compositional process, in which a person who used to take two or more steps to do a problem now only takes one (Blessing et al., 1996). A number of skill acquisition theories have attempted to explain the learning mechanisms underlying compositional processes such as step skipping. Some of these explanations suggest that step skipping involves replacing smaller grain operators with larger grain operators.

One such explanation is provided by Newell's (1990) theory of cognition, SOAR. This explanation makes use of many of the constructs germane to Newell and Simon's (1972) production system framework (e.g., problem space, operators and production rules). The problem space consists of physical states or knowledge states achievable by the problem solver. The problem solving task involves finding a sequence of operators that transform the initial state into a goal state in which the goal is satisfied. Problem solving thus is conceived of as a search of the problem space. An operator is a rule that can be applied to transform the current problem state into a different one. Operators are often conceived of as production rules, which are condition-action (IF-THEN) statements that act on the current problem state. SOAR has a single mechanism to account for all skill acquisition, *chunking*. Chunking operates when an impasse in the problem-solving process occurs. In order to overcome an impasse, SOAR first creates a subgoal to solve the impasse and then creates a new problem space in which to solve the subgoal. Once the subgoal has been solved, SOAR constructs a new operator, a chunk, using the results obtained after the impasse is resolved. The operator that represents the chunk is constructed by using the original elements that occurred before the impasse as the condition side of a production rule and the results after the impasse has been overcome as the action side.

Within the framework provided by SOAR, step skipping can be explained by the construction of larger and larger chunks.

Given the important implications that step skipping has for both learning theory and for teaching, it is rather surprising that few researchers have explored how people learn to skip steps in solving problems (Blessing et al., 1996). Most of the research concerning step skipping has focused on formal, procedurally oriented domains such as "textbook" algebra or physics tasks. Furthermore, the participants in most of these studies have been adults such as undergraduate university students. The generality of the findings from these studies of adults' step skipping behaviour in formal, procedurally oriented domains to young primary school students operating in less formal, conceptually oriented domains thus is open to question.

## THE CURRENT STUDY

In the study being reported in this paper, we investigated young children's step skipping in a less formal, conceptually oriented domain: the solution of partitive quotient fraction problems. This study was conducted in the context of larger study in which we investigated the partitioning strategies utilised by the children when they were asked to solve sequences of "realistic" partitive quotient fraction problems (cf. Streefland, 1991). In the partitive quotient fraction construct, the fraction $^{x}/n$ refers to the quantity represented by one of the resulting shares when the quantity $x$ is partitioned into $n$ equal parts. For example, if two whole cakes are partitioned into three equal shares, then each share is two thirds of a cake. The major findings from the overall study are reported in Charles (1998), and Charles and Nason (submitted).

### Method

The data gathering technique used was the clinical or mixed method technique of Ginsburg, Kossan, Schwartz, and Swanson (1983) which is a combination of Piaget's clinical interview and the talk aloud procedures of Ericsson and Simon (1984). According to Ginsberg et al., this technique enables researchers to not only elicit complex intellectual activity but also to identify the internal symbolic mechanisms underlying the complex intellectual activity.

**Participants:** Twelve third grade students (aged between 7.9 to 8.3 years) from a primary school located in Eastern Australia participated in this study. The children were chosen through serial, contingent, purposeful and exhaustive sampling (Guba & Lincoln, 1989) to produce maximum variation in cognitive functioning. Each participant was chosen after consultation with the students' classroom teacher in order to maximise the probability that as many strategies as possible would emerge during the course of the study. When it was found that no new partitioning strategies or other insights about young children's partitioning emerged during the interviews with the eleventh and twelfth participants, data collection ceased. The size of the sample thus was influenced by Guba and Lincoln's criteria of redundancy.

**Instruments:** A set of thirty realistic partitioning problems were developed for the study. In each of the problems, the children were asked to assume the roles of waiters/waitresses serving pizzas, pancakes, pikelets, icecream bars, apple pies, licorice straps to a number of customers sitting at a restaurant table. An example of one of these problems is presented in Figure 1 below.

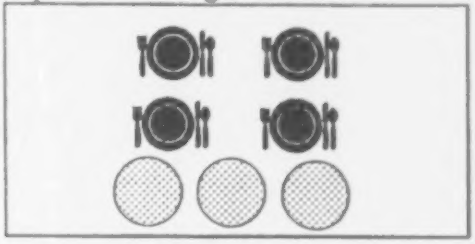


*Figure 1. Partitioning Problem: 3 pizza among 4 people*

Three task variables were taken into consideration during the development of the partitioning problems: (i) types of analog objects, (ii) number of analog objects, and (iii) number of people. The analog objects used were circular region models, rectangular region models and length models. The number of analog objects ranged from one to six and the number of people ranged from two to six. The circular region models for the study included: pizza, pancake, pikelet and apple pie. The rectangular region models included: cake and icecream barcake. The length model was represented by the licorice strap.

**Procedure:** As the literature noted that children have sound informal knowledge of half (Behr, Harel, Post, & Lesh, 1992; Ball, 1993) and powerful strategies for halving (Pothier & Sawada, 1983), each interview began with "pizza" circular region model problems which produced shares of half and quarter. We then proceeded onto rectangular region model and length model problems which produced shares of half or quarter. The sequence of partitioning problems administered to a child after (s)he had attempted these initial problems which generated shares of half or a quarter was not selected a priori. Instead, the interviewer used the child's responses to each problem to inform the selection of problems administered later on in the interview. Each child thus was administered a unique sequence of partitioning problems. However, each child was asked to solve problems involving : 1) all types of analog objects, 2) one to six analog objects being shared, and 3) two, three, four, five or six shares.

In order to produce a more dynamic assessment of the children's knowledge structures, the interviews were often but not always extended to include limited teaching episodes. According to Hunting (1980), the inclusion of limited teaching episodes within clinical interviews takes the interview a step beyond merely assessing the status of a child's cognitive functioning. During these limited teaching episodes, there was no direct teaching of partitioning strategies. The teaching episodes instead focused on: (1) exploring and extending children's knowledge construction, or (2) helping the children to overcome impasses encountered while attempting to solve a partitioning task by asking appropriate focusing questions.

Each of the interview sessions was videotaped and later on transcribed to facilitate the process of data analysis.

**Data analysis:** In order to identify the partitioning strategies employed by the children, the children's protocols were analysed using a grounded theory approach such as that proposed by Strauss and Corbin (1990). We also applied a SOAR production system analysis to the protocols in order to gain insights into the processes underlying the children's step skipping. The results and discussion presented in this paper predominantly focus on the SOAR production system analysis.

## RESULTS AND DISCUSSION

During the course of this study, we noticed that as each of the children progressed through their sequence of partitive quotient fraction problems, they all engaged in step skipping behaviour. Two classes of step skipping were noted. Some of the step skipping was physical in nature. For example, the number of cuts (or partitions) applied to the analog objects were reduced and/or the number of sharing moves were reduced. Some of the step skipping was cognitive in nature. That is, the number of cognitive steps utilised during the solution procedure was reduced.

In order to identify some of the key aspects about the step skipping noted during the course of this study, we will now employ a SOAR production system framework to analyse problem solving episodes from two of the children: Caitlin and Joshua.

**Caitlin's step skipping:** Near the end of her interview session, Caitlin was asked to solve the problem of sharing one licorice strap between four people. In order to solve this problem, Caitlin applied Operator CC1 (**C**aitlin **C**hunk **1**). Operator CC1 states that:

> *IF one length analog model has to be shared between four people,*
> *THEN the length analog model is divided into quarters AND one quarter is given to each person.*

When Caitlin was then asked to solve the problem of sharing three licorice straps between four people, she was faced with an impasse because she had no operator capable of being immediately applied to the existing problem space. In order to solve the impasse, she created a new subgoal of sharing each of the straps individually. The problem space she created to solve the subgoal thus consisted of three problem spaces similar to that which she encountered during the solution of the previous problem (i.e., one strap being divided into four equal shares).

With the new subgoals and problem space, she was able to apply Operator CC1 to overcome the impasse and solve the problem. Her procedure for solving the problem was:

Step 1: Applied Operator CC1 to first strap
Step 2: Applied Operator CC1 to second strap

Step 3: Applied Operator CC1 to third strap
Step 4: Applied a quantifier operator to determine how much in each share.

Following her successful solution of this partitive quotient fraction problem, she was able to abstract the notion that the quantity in each share could be determined by dividing the number of objects to be shared by the number of people sharing. She thus was able to construct a larger grained operator (or chunk) CC2. Operator CC2 states:

*IF there are x analog objects to share between n people,*
*THEN each share is x/n.*

Thus, when she was asked to solve the problems of sharing tenlicorice straps and one hundred licorice straps between four people, she applied Operator CC2. This operator enabled her to skip Steps 1-4 and instantly generate correct answers of 10/4 and 100/4 for each these two problems.

**Joshua's step skipping:** The first problem administered to Joshua was to share one pizza between four people. Joshua applied Operator JC1 (Joshua Chunk 1) and was able to successfully solve the problem. Operator JC1 states that:

*IF one circular region analog model has to be shared between four people,*
*THEN the circular region model analog model is divided into quarters*
*AND one quarter is given to each person.*

Following this, he was administered the problem of sharing two pizza between four people. To solve this problem, he created subgoals of "cutting" one pizza for two people and "cutting" the second pizza for the other two people. The problem space he created to solve the subgoals consisted of two problem spaces each of which contained one pizza to be divided between two people. In each of these new problem spaces, he was able to directly apply a previously acquired operator JC0 which states:

*IF one whole analog object has to be shared between two people,*
*THEN one half of the analog object is given to each person.*

His procedure for solving this problem was:

Step 1: Applied Operator JC0 to first pizza
Step 2: Applied Operator JC0 to second pizza
Step 3: Applied a quantifier operator to determine how much in each share.

By applying Operator JC0 rather than Operator JC1 to this problem, Joshua was able to reduce the number of physical steps involved in completing the problem. If Operator JC1 had been applied, Joshua would have had to carry out four cutting actions (i.e., cutting each whole into quarters) and eight sharing actions (i.e., sharing eight quarters between four people). When JC0 was applied, the cutting actions were reduced to two moves (i.e., cutting each whole into halves) and the sharing actions were reduced to four moves (i.e., sharing four halves between four people).

Although this procedure involving the repeated application of JC0 reduced the number of physical steps and made the generation of equal shares easy, it had one major flaw. Because the shares generated by this procedure were one half rather than two quarters, Joshua failed to abstract the notion of the partitive quotient fraction construct from the solution of the problem (i.e., to construct an operator such as:*IF two wholes are partitioned into four equal shares, THEN each share can be quantified by the fraction 2/4*).

Following the completion of the second problem, Joshua used the process of chunking to create a new operator JC2 which states:

*IF two whole analog objects have to be shared between two people,*
*THEN the two analog objects are divided into four halves*
*AND one half of an analog object is given to each person.*

Joshua was then administered the problem of sharing three pizzas between four people. To overcome the impasse he faced when confronted by this problem, he redefined the problem into two new problem spaces: one in which the subgoal was to share two pizzas between four people and the other in which the subgoal was to share one pizza between four people. To solve the first subgoal, Joshua applied the newly acquired Operator JC2. To solve the second subgoal, he applied Operator JC1. Thus, his procedure for solving this problem was:

Step 1: Applied Operator JC2 to first two pizzas
Step 2: Applied Operator JC1 to third pizza
Step 3: Applied a quantifier Operator to calculate 1/2 + 1/4 = 3/4

Following the completion of this problem, Joshua used the process of chunking to create a new operator JC3 which states:

*IF three whole analog objects have to be shared between four people,*
*THEN the two of analog objects are divided into four halves AND one half of an analog object is given to each person AND one analog object is divided into quarters AND one quarter of the analog object is given to each person*
*AND quantify the share by adding 1/2 and 1/4*

When Joshua was immediately readministered the same problem, he did not apply Operator JC3. Instead, he was instantly able to state that each person should receive 3/4. In addition to this, he minimised his cutting actions by removing one quarter from each of the three pizza. He thus created three intact three quarter sized pieces plus three separate one quarter sized pieces. The three intact three quarter sized pieces were shared between the first three people. The three separate one quarter sized pieces were given to the fourth person.

At first, we thought this indicated that Joshua had by the process of chunking constructed a large grained operator such as Caitlin's Operator CC2. However, when he was readministered this same problem after he had been asked to solve the problem of sharing two pizzas between five people, he regressed back to applying Operator JC3. This regression to Operator JC3 indicated that he had merely recalled the answer to the problem and not constructed a CC2-like operator.

**Discussion:** The SOAR production system analysis of these two children's problem solving episodes revealed a number of important insights about the children's step skipping. First, some step skipping facilitated the construction of larger grained general operators (such as Caitlin's Operator CC2) which enabled the children to instantly generate correct answers for partitive quotient fraction problems. This occurred when the chunking constructed operators which created an overt relationship between the number of analog objects, the number people sharing and the amount in each share (i.e., $x$ objects partitioned into $n$ equal parts produces shares of $x/n$).

Second, some step skipping produced larger grained operators (such as Joshua's JC3) which reduced the number of physical steps involved in the solution of the problems but did not result in the construction of an operator which would enable the child to instantly generate correct answers for partitive quotient problems. Further analysis revealed that this occurred when the chunking failed to construct operators which created an overt relationship between the number of analog objects, the number people sharing and the amount in each share.

Thus, it seems that only some step skipping leads to: 1) making the process of solving problems quicker and easier, and 2) the construction of operators which facilitate the abstraction of the partitive quotient fraction construct from the problem solving activity. Some step skipping only leads to making the physical process of generating equal shares easier.

## SUMMARY AND CONCLUSION

In this study, we reported on the step skipping manifested by two Grade 3 students engaged in the process of solving sequences of partitive quotient fraction problems. Our results tend to confirm findings from previous research studies that step skipping helps learners to solve problems more quickly, efficiently and easily. However, our study also found that step skipping may, if certain conditions are met, help learners to construct operators which facilitate the abstraction of key concepts from the problem solving activity. Conversely, it may hinder the abstraction of key concepts.

Most previous studies of step skipping have been conducted within formal, procedurally oriented domains such as "textbook" algebra. Our investigation of the children's step skipping was conducted within a less formal, conceptually oriented domain. The insights generated during the course of our study seem to indicate that the investigation of step skipping outside the formal, procedurally oriented domains may lead to further important insights about the roles of step skipping in the learning of mathematical concepts and processes.

The SOAR production system framework played a crucial role in identifying: 1) the different types of step skipping, 2) the relationships between the different types of step skipping procedures, and 3) the knowledge structures that underpin the step skipping procedures. It facilitated the generation of explanations of why not all skip stepping necessarily leads to the construction of operators which create overt relationships between the number of analog objects, the number people sharing and the amount in each share. Thus, it played a crucial role in identifying and explicating the processes underlying step skipping. The efficacy of the SOAR production system framework during the course of this study indicates that it could be used most advantageously in future investigations of young children's step skipping within less formal, conceptually oriented domains.

**Implications for teaching:** The results from this study have a number of important implications for teachers. First, teachers need to realise that not all step skipping leads to the abstraction of key constructs. Therefore, they need to carefully monitor the step skipping behaviour of children during problem solving episodes to ensure that it is leading to the construction of powerful operators which facilitate the abstraction of key concepts. Furthermore, they need to realise that if the children's step skipping is not leading to the construction of these powerful operators, modifications need to be made to the children's sequences of learning activities.

## REFERENCES

Ball, D.L. (1993). Halves, pieces and tooths: Constructing and using representational contexts in teaching fractions. In T.P. Carpenter, E.Fennema, & T.A. Romberg (Eds.) *Rational Numbers: An integration of research* (pp. 157-195). Hillsdale NJ: Lawrence Erlbaum.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Blessing, S.B., & Anderson, J.R. (1996). How people learn to skip steps.*Journal of Experimental Psychology, 22*(3), 576-598.

Charles, K.L. (1998). *Towards the specification of a computer system to facilitate the learning of the partitive quotient fraction construct.* UnpublishedPhD thesis, Queensland University of Technology.

Charles,K.L., & Nason, R.A. (submitted). The abstraction-ability of young children's informal partitioning strategies. *Educational Studies in Mathematics.*

Ericsson, K.A., & Simon, H.A. (1984). *Protocol analysis: Verbal reports as data.* Cambridge, MA: MIT Press.

Ginsburg, H., Kossan, N., Schwartz, R., & Swanson, D. (1983). Protocol methods in research on mathematical thinking. In H.P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 7-47). New York: Academic Press.

Guba, E., & Lincoln, Y. (1989). *Fourth generation evaluation.* London: Sage.

Hunting, R.P. (1980). *Emerging methodologies for understanding internal processes governing children's mathematical behaviour.* Paper presented at the Fourth Annual Conference of the Mathematics Education Research Group in Australasia, Hobart.

Koedinger, K.R., & Anderson, J.R. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Sciience, 14*(4), 511-550.

Newell, A. (1990). *Unified theories of cognition.* Cambridge, MA: Harvard University Press.

Newell, A., & Simon, H. (1972). *Human proving solving.* New York: Prentice Hall.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics, 14*, 307-317.

Streefland, L. (1991). *Fractions in realistic mathematics education.* Dordrecht: Kluwer Academic.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* London: Sage.

# Making, Having and Compressing Formal Mathematical Concepts

**Erh-Tsung Chin** & **David Tall**

Mathematics Education Research Centre
University of Warwick, UK
e-mail: E-T.Chin@warwick.ac.uk, David.Tall@warwick.ac.uk

*This paper studies the mathematical concept development of novice university students introduced to formal definitions and formal proof, with empirical data collected on "equivalence relation" and "partition". Before meeting the formal definitions of these concepts, the student will already have informal knowledge that allows some intuition as to their meaning. Our focus is on how the concepts are given a formal basis to become part of a formal theory. At first the definitions themselves will be used to establish that certain concepts satisfy the definitions and that other properties can be deduced from them. Later the statements of the theorems may be used to relate concepts without necessarily unpacking them. In this paper we consider responses to a questionnaire and categorise them as various kinds of informal and formal argument.*

The introduction of formal proof in mathematics involves a significant shift from the computation and symbol manipulation of elementary arithmetic and algebra to the use of formal definitions and deduction. This shift alters the way in which language is used from an everyday informal register to a formal mathematical register (in the sense of Halliday, 1975), recently described by Alcock and Simpson (1999) as the "rigour prefix". This changes register from informal "loosely" speaking to formal "strictly speaking" in mathematics. The shift from informal to formal thinking is by no means easy. Intuitive deduction occurs using "embodied arguments" that build on bodily sensations, such as "inside inside" is "inside" (Lakoff & Johnson, 1999, p.32). Thus a formal statement such as "$A \subset B$, $B \subset C$ implies $A \subset C$" is simply "obvious". Everyday arguments often proceed by using prototypical examples referring to categories that have fuzzy boundaries (eg "short people") in the sense of Rosch (1973) and Lakoff (1987). Lakoff and Johnson argue that all thought, including formal deduction, is based on embodied conceptions, so that formal thought also includes informal elements.

The shift from informal to formal thought in mathematics does not occur in a single step. At first the mathematics is "definition-based" where proofs relate back to the fundamental definitions. It then becomes increasingly "theorem-based" as new proofs are based on previously established theorems that are no longer unpacked to the original definitions. Although the mathematician may see this as a simple programme of building up a formal theory, the student is faced with considerable cognitive reconstruction. First there is the shift from informal concepts—that already "exist" in the mind of the student and may be described verbally—to formal concepts in which the definition is given and the concepts must be constructed by deduction. Cognitively the concepts need to grow in interiority (Skemp, 1979), and become compressed (Thurston, 1990) so that they may be used imaginatively and efficiently. Even at this level, informal mental images of concepts such as "partition" may be used side-by-side with formal concepts.

In this paper we consider the cognitive growth of "equivalence relation" and "partition" at a time when students have been given the definitions and have been expected to operate in an increasingly "theorem-based" manner. We analyse whether they still operate informally, whether they are using definitions and theorems formally, or whether they have compressed the ideas of equivalence relation and partition as a single flexible cognitive unit (in the sense of Barnard & Tall, 1997).

### Making and Having (Informal or Formal) Concepts

The shift to the formal mathematical register is a continual development in which the status of concepts becomes increasingly rich and formal. A formal course presents a sequence of theorems:

Theorem 1, Theorem 2, ..., Theorem $N$, ...,

interspersed with new definitions introduced as the theory becomes more extended. The purpose of Theorem $N$ is to deduce the properties to be proved in its statement using the axioms, definitions and previous theorems 1, 2, ... , $N-1$. For the learner, a concept that needs to be proved one day becomes a concept that can be used without proof the next. Pirie and Kieren (1994) formulated the distinction between *making* images and *having* images. The formal course essentially expects the students to be *making concepts* during the proof activity, and then *having* these concepts for future development. In practice, students rarely "have" the concepts of definitions or axioms in a form that can be utilised formally as they prepare to "make" the next idea formal. Bills & Tall (1998) found that in developing the notion of "least upper bound" most of the students interviewed did not have an operable grasp of the formal definition during the ensuing parts of the course that implicitly required it. There were, however, instances of a student able to make use of an informal understanding as part of what appeared to be a formal proof.

### Definition-based Formal Mathematics

Initially formal mathematical concepts are given in terms of definitions. For example, an equivalence relation may be defined as follows:

> An *equivalence relation* on a set $S$ is a binary relation on $S$ that is
> *reflexive*: $a \sim a$ for all $a \in S$
> *symmetric*: if $a \sim b$ then $b \sim a$ for all $a, b \in S$
> and *transitive*: if $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in S$. (Stewart & Tall, 1977)

This illustrates several difficulties faced by students being presented with formal definitions. To understand it, the student must already "have" the notion of "set" $S$ and of "binary relation" on $S$. However, the first of these is not (and cannot be) given a formal definition at this stage. The second can be given informally using the informal notion of relation between two things which either holds or does not. It can also be introduced as a function from $S \times S$ to $S$, which now requires the notions of *cartesian product* $S \times S$ and *function*, each of which (especially the latter) has subtle cognitive difficulties (Sierpinska, 1992). The steady accumulation of concepts based on both informal and formal ideas can lead to a feeling of uneasiness in attempting to deal with them.

### Theorem-based Formal Mathematics

Under this heading we consider those deductions that use the *results* of theorems without necessarily going right back to the definitions themselves. This occurs increasingly as more and more formal concepts are introduced. For example, a bijection is defined as follows:

> A function $f: A \to B$ is a *bijection* (or is a *one-to-one correspondence*) if it is both an injection and a surjection (to B). (Stewart & Tall, 1977)

Notice again that this definition requires the student to "have" the concept of injection and surjection, which in turn depend on the concept of function. However, soon after the definition is made, the following theorem is proved:

> The *identity* map is a bijection.
> The *composition* of bjections is a bijection.
> The *inverse* of a bijection is a bijection.

This theorem may be used in solving the following:

> A relation on a set of sets is obtained by saying that a set $X$ is related to a set $Y$ if there is a bijection $f: X \to Y$. Is this relation an equivalence relation?

A definition-based deduction uses the original definitions of concepts. A theorem based deduction refers to theorems (in this case usually the three separate components, each matching one of the three parts of the definition of equivalence relation).

### Compressed Concept-based Mathematics

Some students use their knowledge of concepts in a much more flexible and imaginative way, for instance by identifying the notion of equivalence class directly its corresponding partition within a single cognitive unit, enabling a given problem to be approached by linking directly to whichever properties are required at a given time.

### Empirical Study

Thirty six students taking mathematics at one of the top five universities in the UK responded to questionnaires, 18 from a course for mathematics majors and 18 others taking mathematics in a course such as statistics or economics. Both courses covered the same material over a ten-week term with three lectures per week. The questionnaire was given out six weeks after the definitions of equivalence relation and partition had been formulated, with the subsequent time used to develop the formal theory. Two questions invited the students to say what they understood by given concepts, two more investigated the use of a definition, one in an informal context, the other in a formal context that could also involve definitions, theorems, or an alternative insightful view:

1. Say what "equivalence relation' means to you.
2. Say what "partition" means to you.
3. If $M$ is the set of all mathematics students at Warwick, is the relation "has the same surname as" an equivalence relation?
4. A relation on a set of sets is obtained by saying that a set $X$ is related to a set $Y$ if there is a bijection $f: X \to Y$. Is this relation an equivalence relation?

### Student Responses

The student responses to the first two questions were analysed to see if the students gave some kind of operable definition or not. Definitions were classified as:

*Formal/detailed*: giving an "essentially correct" formal definition in full detail,
*Informal/outline*: either an informal verbal description, or "reflexive, symmetric, transitive",
*Example*: giving a single specific or general example,
*Picture*: using visual imagery in a drawing.

For instance, the following student gave an outline response to 1 and an example for 2:

**Say what "equivalence relation" means to you:**

A relation which is reflexive, symmetric and transitive.
Of course. What else?

**Say what "partition" means to you:**

$$\left\{ \begin{array}{l} A \cap B = \emptyset \\ A \cup B = C \end{array} \right\} \Rightarrow A, B \text{ is a partition on } C.$$

In question 1 the majority of students were able to give definitions, although many were informal, or simply specified "reflexive, symmetric, transitive", (table 1).

| | Mathematics Majors (N=18) | Others (N=18) | Total (N=36) |
|---|---|---|---|
| Formal/detailed | 5 | 6 | 11 |
| Informal/outline | 8 | 11 | 19 |
| ***Total definitions*** | *13* | *17* | *30* |
| Example | 1 | 0 | 1 |
| Picture | 0 | 0 | 0 |
| Others | 4 | 1 | 5 |
| No response | 0 | 0 | 0 |

Table 1: Responses to "equivalence relation"

Fewer students offered a definition of a partition, with less than half in the non-mathematics majors, (table 2).

| | Mathematics Majors (N=18) | Others (N=18) | Total (N=36) |
|---|---|---|---|
| Formal/detailed | 6 | 5 | 11 |
| Informal/outline | 8 | 2 | 10 |
| ***Total definitions*** | *14* | *7* | *21* |
| Example | 1 | 1 | 2 |
| Picture | 0 | 2 | 2 |
| Others | 2 | 6 | 8 |
| No response | 1 | 2 | 3 |

Table 2: Responses to "partition"

Question 3 was formulated in an informal context to see how the students would respond using the formal notion of "equivalence class", (table 3).

| | | Mathematics Majors (N=18) | Others (N=18) | Total (N=36) |
|---|---|---|---|---|
| *Informal* | Informal Definition | 5 | 11 | 16 |
| | Other | 1 | 1 | 2 |
| | No response | 0 | 1 | 1 |
| *Formal*<br>*perhaps with some informal language* | Definition | 9 | 4 | 13 |
| | Theorem | 0 | 0 | 0 |
| | Partition | 3 | 1 | 4 |

Table 3: Responses to the informal "surnames" question

Sixteen out of thirty six were classified as operating informally; they either reproduced the definition with no reference to the problem (eg *a~a*; *a~b⇒b~a*; *a~b, b~c ⇒a~c*) or they responded in an informal prototypical manner:

refl : Smith = Smith ✓
trans: Smith (A's) = Smith (B's) = Smith (C's) ✓
symm: Smith (A's) = Smith (B's) $\Leftrightarrow$ Smith (B's) = Smith (A's) ✓

Thirteen responded in a more formal manner, using set theoretic formalism, eg:

✓① $\forall m \in M$ $(m,m) \in \sim$ $\Rightarrow$ You have the same surname as you. TRUE
✓② If $(m,n) \in \sim$ then m and n have the same surname
If $(n,p) \in \sim$ then n and p have the same surname
So m and p must have the same surname $\Rightarrow (m,p) \in \sim$
✓③ If $(m,n) \in \sim$ then m has the ... as n $\Leftrightarrow$ n has the ... as m $\Rightarrow (n,m) \in \sim$

No responses used theorems (because the problem focused on the use of the definition), however, four responded in terms of partitions:

This partitions the set of maths students into sets containing students with the same surname. As there is a bijection between partitions and equivalence relations, this too is an equivalence relation.

Question 4 revealed a wider spectrum of responses, (table 4).

Twenty one gave some kind of informal response, including those who simply wrote down an outline definition:

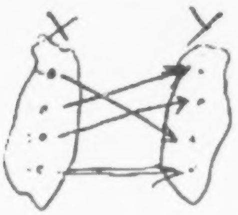
refl symm trans

BEST COPY AVAILABLE

| | | Mathematics Majors (N=18) | Others (N=18) | Total (N=36) |
|---|---|---|---|---|
| *Informal* | Informal Definition | 7 | 8 | 15 |
| | Other | 2 | 3 | 5 |
| | No response | 0 | 1 | 1 |
| *Formal* *perhaps with some informal language* | Definition | 2 | 0 | 2 |
| | Theorem | 5 | 6 | 11 |
| | Partition | 2 | 0 | 2 |

Table 4: Responses to the formal "bijection" question

Others were unable to make sense of the question, eg:



No, not necessarily as x~x thr doesn't always happen.

Only fifteen gave a formal response (often using at least some informal language). Of these, eleven were *theorem-based*, using the parts of the theorem concerning bijections:

Identity bijection $X \to X$ so $X \sim X$

$f\,X \to Y$, $g\,Y \to Z$ bijections $\Rightarrow \exists$ bijection $X \to Z$ ie $g \circ f$

So $X \sim Y$, $Y \sim Z \Rightarrow X \sim Z$

$X \sim Y \Rightarrow \exists\, f\; X \to Y$ bijection so $f^{-1}\; Y \to X$ is also bijection

So $X \sim Y \Rightarrow Y \sim X$

Of these eleven, only three explicitly mentioned the identity to establish the "reflexive" property, the others only asserting, at most, that there *is* a bijection. This is a most interesting phenomenon worthy of further study. The students seem to be more comfortable giving a *general* argument than using a specific case.

Just two students were *definition-based*, referring back to the definition of bijection:

There is a bijection mapping each element of X to itself Therefore reflexive

If every element of X maps to a unique element of Y, then each element of Y maps to a unique element of X Therefore symmetric

If every element of X maps to a unique element of Y, and every element of Y maps to a unique element of Z, then every element of X maps to a unique element of Z, therefore transitive

Neither of these referred explicitly to the identity function to establish reflexivity.

BEST COPY AVAILABLE

Two students were classified as compressed concept-based, using the notion of partition as being equivalent to the notion of equivalence relation. One explicitly referred to the theory of cardinal numbers, although his response was not fully formal:

> This partitions the set of sets into sets which contain sets with
> the same cardinality (ie have a bijection between them). Thus, as
> this is a partition it is also an equivalence relation on cardinal numbers.

This particular student shows great flexibility in response, treating the concepts of equivalence relation and partition as a single cognitive entity, using whichever properties were appropriate at the time (eg in another question, not given here, he responded in terms of properties of equivalence relations). His definitions of equivalence class and partition were a model of structured precision. He defined equivalence class first at an outline level, then gave the full detail; he described a partition first in a precise verbal way and then repeated it in strict symbolism. He seemed able to operate at a level of his choice, preferring to work at a high conceptual level with the detail implicitly subsumed in his arguments. This is an example of someone working with the concepts concerned compressed as a single highly-connected cognitive unit.

## Summary

The responses to the questionnaire reveal that after the students have been working formally with the notions of equivalence relation and partition for six weeks, more than half of them offered only *informal* responses. Less than half gave *formal* responses in terms of *definitions* or *theorems*. Four others gave broader conceptual responses to question 3, falling to 2 students in question 4 (the other two giving formal theorem responses). This confirms a picture in which the majority of students following a formal course at a highly rated university responded at an informal level after several weeks' experience of formalism. At the same time, two able students worked in a different way using the *compressed concept* that encompassed both equivalence relation and partition.

These findings relate closely to other theories. The definition/theorem approaches use *analytic* powers to *deduce* properties, the formal global category uses *synthetic* powers to *relate ideas together*. This is consonant with the theory of *extracting meaning* (from the definitions) and *giving meaning* (to the concepts) as formulated by Pinto & Tall (1999). It also relates to the process/object phenomena identified by Dubinsky and his colleagues in his APOS theory approach (eg. Asiala, Dubinsky *et al.*, 1997).

The research of Moore (1994) formulates a framework based on "definition - image - usage" and gives many fascinating insights into the usage of images and definitions in formal mathematics. In a sense his distinction between the use of images and the use of definitions has similarities with our focus on informal and formal thinking. However, his paper uses an interpretation of "concept image" which contrasts definition and image as distinct entities. For us the concept image *includes* the definition and its resulting related imagery. This allows us to formulate an ongoing change of the total concept image that steadily builds up the formal register.

BEST COPY AVAILABLE

Our research instrument—a single questionnaire applied at one point in a development —is too restricted a tool to give answers to other questions, in particular, whether there is a hierarchy running through the given categories. The evidence of Pinto & Tall (1999)—taken from an analysis course studied in the same institution—suggests that there is a spectrum of approaches. Some students do not go beyond the informal stage, some go through the stages in the given sequence as a hierarchy by *extracting* meaning from the definition. Others perform thought experiments from the very beginning, building up their theoretical perspective by modifying their images and *giving* meaning to the definition and its subsequent deductions.

This research, together with other sources mentioned in this paper, shows the difficulty of building the formal register in a first university course in mathematics. Perhaps matters could be improved by explicitly encouraging students to gain an overall view of the strategies involved in the transition to the formal register. However, using the memorable phrase of Sfard, (1991) this may involve a "vicious circle" where the strategy to understand formal proof is difficult to comprehend until the student has experienced formal proof itself. This learning strategy remains an investigation for another time.

## References

Alcock, L. & Simpson, A. (1999). The rigour prefix. In O. Zaslavsky (ed.), *Proceedings of the $23^{rd}$ Conference of the International Group for the Psychology of Mathematics Education*, 2, 17–24.

Asiala, M., Dubinsky, E., Mathews, D. M, Morics, S. & Okta, A. (1997). Development of students' understanding of cosets, normality, and quotient groups, *Journal of Mathematical Behavior*, 16, 241–309.

Barnard, A. D. & Tall, D. O. (1997). Cognitive units, connections, and mathematical proof. In E. Pehkonen, (Ed.), *Proceedings of the $21^{st}$ Annual Conference for the Psychology of Mathematics Education*, 2 , 41–48.

Bills, L. & Tall, D. (1998). Operable definitions and advanced mathematics: the case of the least upper bound, *Proceedings of the $22^{nd}$ Conference of the International Group for the Psychology of Mathematics Education* 2, 2, 104–111.

Halliday, M. (1975). Some aspects of socio-linguistics , *Interactions between Linguistics and Mathematical Education*, UNESCO, Copenhagen, 64–73.

Lakoff, G. & Johnson, M. (1999), *Philosophy in the Flesh*, New York: Basic Books.

Lakoff, G., (1987), *Women, Fire and Dangerous Things*, Chicago: University of Chicago Press.

Moore, R. C. (1994). Making the transition to formal proof, *Educational Studies in Mathematics*, 22, 249–266.

Pinto, M. & Tall, D. O. (1999). Student constructions of formal theory: Giving and extracting meaning. In O. Zaslavsky (ed.), *Proceedings of the $23^{rd}$ Conference of the International Group for the Psychology of Mathematics Education*, 4, 65–72.

Pirie, S. E. B. & Kieren, T. E. (1994). Growth in mathematical understanding: how can we characterise it and how can we represent it? *Educational Studies in Mathematics*, **26** (3), 165–190.

Rosch, (1973), Natural Categories, *Cognitive Psychology*, 4, 328 – 350.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics, 22*, 1–36.

Sierpinska, A. (1992). Theoretical perspectives for development of the function concept. In G. Harel & E. Dubinsky (eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA Notes 25, (pp.23–58). Washington DC: MAA.

Skemp, R. R. (1979). *Intelligence, Learning, and Action*. New York: John Wiley & Sons.

Stewart, I. N, & Tall, D. O. (1977). *The Foundations of Mathematics*, Oxford University Press .

Thurston, W. P. (1990). Mathematical Education, *Notices of the American Mathematical Society*, 37 7, 844–850.

BEST COPY AVAILABLE

# Mental Projections in Mathematical Problem Solving: Abductive Inference and Schemes of Action in the Evolution of Mathematical Knowledge

Victor V. Cifarelli
Department of Mathematics
The University of North Carolina at Charlotte

*Combining aspects of Piaget's scheme theory and Peirce' theory of abduction, this paper examines the novel problem solving actions of a college student. The analysis documents and explains the important role of abductive inference in the solver's novel solution activity.*

## Introduction

Abstraction and abduction describe creative processes in mathematical problem solving. The solver's ability to abstract mathematical relationships from their problem solving actions enables them to create mathematically powerful ideas (Schoenfeld, 1985). The process of abduction (as described by Charles Saunders Peirce), wherein explanatory hypotheses are generated and tested, enables solvers to scrutinize their potential problem solving activity and make conjectures about its ultimate usefulness (Anderson, 1995; Cifarelli, 1998; Fann, 1970; Mason 1995).

The purpose of this paper is twofold: (1) to document the importance of abduction as a key process of mathematical problem solving, and (2) to demonstrate how a focus on the abductive reasoning activities of problem solvers enhances and extends contemporary constructivist analyses, which rely on abstraction to explain the solver's construction of new knowledge. The first part of the paper provides a brief overview of the theories of Peirce and Piaget, focusing on how each explains the construction of new knowledge as involving acts of problem solving. The second part of the paper focuses on a Piagetian study of problem solving, previously conducted by the author. Through re-examination of selected episodes of problem solving activity, the revised analysis demonstrates the prominent role that abduction plays in problem solving activity and shows how the Peircean analysis enhances and extends the Piagetian analysis in explaining novel solution activity.

## Using Peircian and Piagetian Perspectives to Study Problem Solving

While the theories of Peirce and Piaget are not usually viewed as problem solving theories, each of these scholars had high regard for problem solving activity and its prominent role in the evolution of new knowledge. The central focus of each of their theories, Peirce's focus on the importance of logical reasoning individuals use to explain unexpected or surprising facts, and Piaget's focus on how a learner's thinking proceeds in the face of cognitive perturbation, suggests they each saw a fundamental relationship between problem solving and learning: when an individual solves a problem, they have engaged in learning activity and have constructed new knowledge.

Peirce cited the importance of ***abduction*** (the generation of plausible hypotheses to account for surprising facts) as the process that introduces new ideas into the reasoner's actions. It is the initial proposal of a plausible hypothesis on probation to account for the facts.

In contrast to Peirce's views about reasoning, Piaget maintained a structural view of knowledge in developing his ideas on epistemology. According to Piaget, learners actively organize their sensori-temporal actions into mental structures, or ***schemes***, which can be evoked to aid the learner's interpretive acts when problems are encountered. Through repeated application, schemes become ***operative*** as they are generalized and extended.

## Further Comparisons Between Piaget and Peirce

Both Peirce and Piaget viewed the construction of new knowledge as involving dynamic, creative activity (Table 1). Peirce argued that abduction, through the generation of novel hypotheses, contributes to creative acts that may lead to new knowledge; alternatively, Piaget explained the process in terms of abstraction, describing reflective abstraction as the primary creative process that explains the re-organization of action that takes place when schemes are revised.

**Table 1: Problem Solving-Based Explanations of New Knowledge**

| | ***Peirce*** | ***Piaget*** |
|---|---|---|
| *constructs* | abductive inference | reflective abstraction |
| *key processes* | hypothesis generation and testing: plausible hypotheses are self-generated and tested to explain surprising results | through resolution of perturbations, learners construct and reconstruct their knowledge at increasingly abstract levels |
| *new knowledge* | hypotheses evolve through intertwining of induction, deduction, and further abductions | learners develop structure in their problem solving activity (schemes of action); schemes are generalized through repeated application and become operative (anticipation) |
| *growth of awareness* | explanatory hypotheses | anticipation |

While the differences stated above amount to different emphases regarding what should count as meaningful problem solving, I believe the differences are viewed more profitably as Peirce and Piaget incorporating fundamentally different ideas about how the learner projects their ideas through time and space. The following sections analyze the problem solving actions of a college student named Marie. Marie was a subject in an earlier study conducted by the researcher, using a Piagetian framework to examine problem solving activity. By re-examining critical junctures of her solution activity using a Peircian lens, the revised analysis will shed light on the differences cited above.

## Marie's Problem Solving

Marie was interviewed as she solved a set of nine similar algebra word problems. These problems were designed by Yackel (1984) to induce problematic situations (Table 2) and provided opportunities for Marie to compare and contrast her actions across a range of similar problem solving situations and hence, develop her ideas about problem "sameness" (Lobato, 1996) in the course of her on-going activity.

| **Table 2: Sample of Algebra Word Problems Used in the Study** |
|---|
| **TASK 1: Solve the Two Lakes Problem**<br>The surface of Clear Lake is 35 feet above the surface of Blue Lake. Clear Lake is twice as deep as Blue Lake. The bottom of Clear Lake is 12 feet above the bottom of Blue Lake. How deep are the two lakes? |
| **TASK 2: Solve a Similar Problem Which Contains Superfluous Information**<br>The northern edge of the city of Brownsburg is 200 miles north of the northern edge of Greenville. The distance between the southern edges is 218 miles. Greenville is three times as long, north to south as Brownsburg. A line drawn due north through the city center of Greenville falls 10 miles east of the city center of Brownsburg. How many miles in length is each city, north to south? |
| **TASK 3: Solve a Similar Problem Which Contains Insufficient Information**<br>An oil storage drum is mounted on a stand. A water storage drum is mounted on a stand that is 8 feet taller than the oil drum stand. The water level is 15 feet above the oil level. What is the depth of the oil in the drum? Of the water? |
| **TASK 4: Solve a Similar Problem In Which the Question is Omitted**<br>An office building and an adjacent hotel each have a mirrored glass facade on the upper portions. The hotel is 50 feet shorter than the office building. The bottom of the glass facade on the hotel extends 15 feet below the bottom of the facade on the office building. The height of the facade on the office building is twice that on the hotel. |
| **TASK 9: Make Up a Problem Which has a Similar Solution Method** |

The Piagetian analysis of Marie's activity can be summarized as follows. She was inferred to have constructed a structure from her solution activity while solving Task 1. While solving Tasks 2-9, Marie's sense of "problem sameness" evolved to the extent that she could begin to reflect on and *anticipate* results of potential solution activity prior to carrying it out with paper-and-pencil. This development of her solution activity was interpreted as she having constructed a conceptual scheme that enabled her to see each successive task as "the same" and act accordingly to solve the problems (Figure 1). Her growing sense of awareness of the efficacy of her solution activity was characterized in terms of increasing levels of abstraction of the scheme (Table 3).

**Figure 1: Marie's Evolving Scheme**

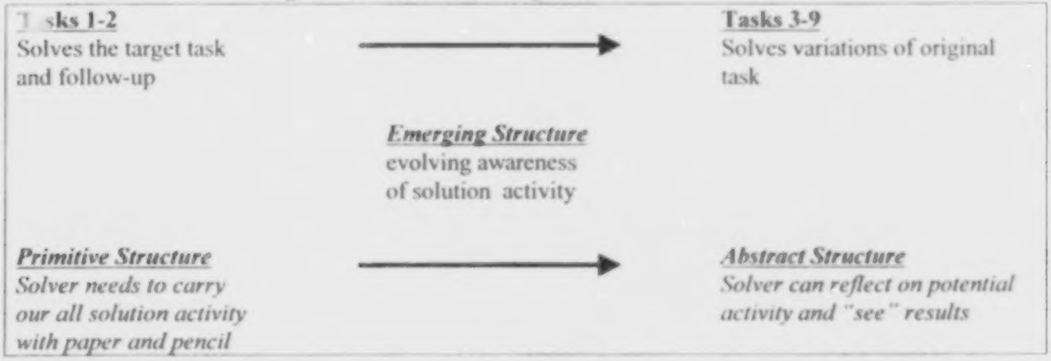

Tasks 1-2
Solves the target task and follow-up
Tasks 3-9
Solves variations of original task
Emerging Structure
evolving awareness of solution activity
Primitive Structure
Solver needs to carry our all solution activity with paper and pencil
Abstract Structure
Solver can reflect on potential activity and "see" results

**Table 3: Marie's Solution Activity as Levels of Abstraction**

| ***Level of Activity*** | ***Characterization*** | ***Examples*** |
|---|---|---|
| ***Abstraction*** | Solver can coordinate potential actions and "run through" *potential* solution activity in thought and operate on its results. | Solver can "see" or *anticipate* results of potential activity (and draw inferences) prior to carrying out solution activity with paper-and-pencil. |
| ***Re-Presentation*** | Solver can coordinate prior actions and "run through" *prior* solution activity in thought. | Solver can "see" or *anticipate* potential difficulties in new problem situations. |
| ***Perceptual Expression*** | Solver uses diagram to aid reflection. | Solver can reason from diagrams to *anticipate* potential problems. |
| ***Recognition*** | Solver sees the relevance of using previously constructed solution activity to solve new problems | Solver *recognizes* usefulness of diagrammatic analysis used to solve Task 1 to solve Tasks 2-9 |

## A Revised Analysis of Marie's Solution Activity

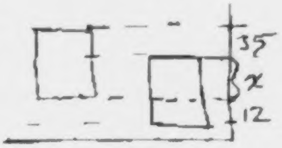
The following sections include episodes from Marie's solution activity. Marie's solution activity while solving Tasks 1 & 2 will be summarized. Her subsequent solution activity in Tasks 3 & 4 will illustrate and explain the gradual generation of novelty into her evolving solution activity in terms of abductive reasoning. The analysis indicates the crucial role abduction plays both as a problem solving process as well as a process that generates new questions and problems.

Marie's solution to Task 1. Marie's solution activity for Task 1 was by no means routine. She initially interpreted the task about the two lakes as an "algebra word problem in two variables" and generated a system of several equations, no two of which were consistent (Figure 2). When she realized that this approach did not lead to a solution, she constructed a side-by-side diagram of the lakes, and translated relevant lengths from the diagram to a vertical axis which served as a reference aid in constructing relationships. This solution activity eventually led to a correct solution (Figure 3).

**Figure 2: Marie's Initial Equations for Task 1**

$$S_c - S_b = 35$$
$$B_c - B_b = 12$$
$$(S_c - B_2) = 2(B_c - B_0)$$

**Figure 3: Marie's Solution to Task 1**



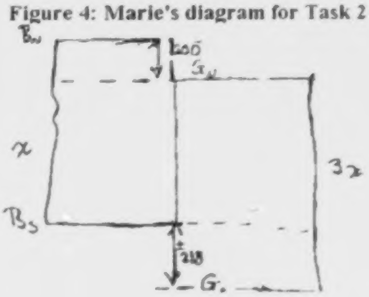
$$35 + x = 2(12 + x)$$
$$35 + x = 24 + 2x$$
$$11 = x$$
$$C = 35 + 11 = 46$$
$$B = 23$$

After solving Task 1, Marie interpreted Task 2 as similar to Task 1, remarking that "the first thing that strikes me is that this problem is a lot like the first one. So I will draw a diagram" and constructed a diagram similar to the diagram she constructed to solve Task 1 (Figure 4). While her anticipation indicated some sense of similarity between solution of the current task and her solution of Task 1, it did not allow her to see and address the potentially problematic information. It was only after she carried out her solution activity that she realized the potentially problematic information, and sensed that the extra information had no bearing on constructing a solution:

> ***Marie***: *This information seems to have nothing to do with the problem. So, I'll just consider all of the other relationships first.*

Marie went on to construct a correct solution for Task 2.

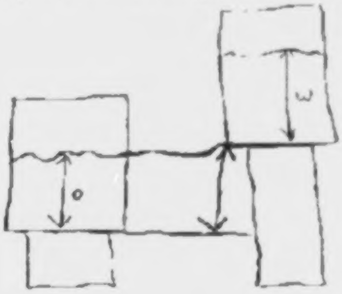
Figure 4: Marie's diagram for Task 2



Marie attempted to solve Task 3 in the same way as she solved Tasks 1 & 2. However, she soon found herself faced with a problematic situation she had not anticipated, at which time she became galvanized with a sense of excitement and wonderment, aggressively looking to explain this new problem she had not anticipated (Figure 5):

> ***Marie***: *I am going to draw a picture. Here is my oil stand. And we have a water storage 8 feet taller. And here's level water. And here's the oil level.* **(reflection)**[1] *So, solve it ... the same way.* **(She smiles, then displays a facial expression suggesting sudden puzzlement)** *Impossible!! It strikes me suddenly that there might not be enough information to solve this problem. I suspect I'm going to need to know the height of one of these things* **(points to containers)**. *I don't know though, so I am going to go over, all the way through.*

Figure 5: Marie's Diagram for Task 3



Marie's anticipation that "the same way" would not work was followed by her *abduction* that the problem might not contain enough information, later refined to the hypothesis that she needed more information about the relative heights of the unknowns. While her hypothesis contained an element of uncertainty, it nevertheless helped her organize and structure her subsequent solution activity, whereupon she explored and tested its plausibility as an explanatory device. Marie spent much time and energy pursuing the elusive information and finally concluded that the problem, as stated, could not be solved.

Marie's solution activity while solving Task 3 suggested a qualitatively different level of inquiry than she demonstrated while solving Tasks 1 and 2. First, the surprise she experienced when she realized her initial anticipation was unconfirmed differed dramatically from the relatively minor surprise she experienced when she encountered the extra information while solving Task 2.

---

[1] **Comments in boldface describe the non-verbal actions of the solver as inferred by the researcher.**

While she could temporarily suspend the conditions of the problem to solve Task 2 (by ignoring the extra information), Marie's sudden experience of surprise while solving Task 3 fueled her desire to generate and entertain novel explanations that were radically different from her prior sense-making actions; this indicated a major opening-up of her conceptual boundaries. She ventured to explore her ideas and convictions with a sense of open-endedness, not particularly concerned for where they might lead, and free in the sense that she no longer was constrained by the conventions she previously operated within. The crucial point here is that her abductive actions opened up conceptual boundaries for potential action, and did not merely suspend the constraining conditions that constituted the problem (as was the case in Task 2).

A second indication that the solver had transformed her actions to a new level of inquiry was the shift in her *reflective orientation*, whereby she began to formulate goals for action in terms drawing from *potential states* of the problem. This change of orientation from present to future events came out of her need to explain a result in "present time" for which there was no room for explanation given her current understandings. With her abduction she generated plausible explanations within the world of future events and imagined action, thereby forging her deliberation over future events that would ultimately served to constrain her current actions. Specifically, she organized her sense-making actions in terms of future events (specific actions concerning the problem conditions she needed to perform in order to verify that a solution was possible) which then *beckoned* back to the solver to make them real. This drawing from the future to chart a present course of action helped the solver make-sense of her problem and also opened up her conceptual horizons so that she could pose new problems (e.g., the problem of designing specific actions that conform to her hypothesis).

In what ways did Marie's abductions help to evolve her solution activity while solving later tasks? A partial answer to this question is that she became more cautious in her activity, spending increased time reflecting on her potential activity. However, her reflections on potential solution activity continued to exhibit hypothetical qualities as she generated novel conjectures. For example, while solving Task 4, Marie quickly noticed the omission of a question from the problem statement yet was able to hypothesize potential problems for her to solve from the information.

***Marie*:** *There's no question!*

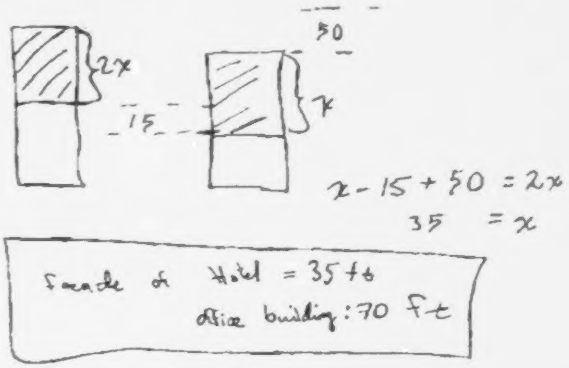
***Interviewer*:** *Is there a problem to solve?*

***Marie*:** **(*Long reflection here*)** ... *The things they could ask for are things like* ... (***HYPOTHESIS***) ... *the height of one of the buildings but* ... (***ANTICIPATION***) ... *there's not enough information to get that*.... *The only thing we have information about is* ... (***HYPOTHESIS***) ... *Ah, the relative heights of the two facades. So, if I were* ... *if somebody wanted me to solve any problem, that's probably what they're asking for.*

Marie's hypothetical statements about potential problems that could be solved were provisional in the sense that they lent themselves to further scrutiny, and plausible since, based on her current understandings, these were problems that could conceivably be solved. Marie's anticipation following her first hypothesis indicated she had deduced from her hypothesis the necessary conditions of the problem, and had performed a mental "run through" of the imagined action of trying to solve the problem, the result of which she rejected her hypothesis. In a similar way she explored the plausibility of her second hypothesis and concluded that it made more sense to her that the problem of finding the heights of the two facades was a problem that could be solved.

Unlike her solution activity in Tasks 2-3, where her solution activity involved making explicit comparison to Task 1 to see how they were similar, here Marie employed hypothetical states of new and future problems in initiating her solution activity. Furthermore, her anticipations were now connected to specific hypotheses. The solver demonstrated this highly abstract activity prior to constructing her diagrams of the two buildings. She proceeded to construct a solution to the problem she had generated, utilizing diagrams to construct relationships, in much the same way as she had done to solve earlier tasks (Figure 6).

**Figure 6: Marie's Solution to Task 4**



***Marie:*** *Okay. Let's see if there is anything here that will at least give me information.* ***(Long reflection here)*** *Okay, the hotel is 50 feet shorter than the office building. So we have distance here which is 50. The facade of the hotel extends 15 feet below the facade of the office building. That distance would be 15. The height of the facade on the office building is twice that on the hotel.* ***(Long reflection here)*** *So I call this distance X, this distance here is 2X. All right!*

***Marie:*** *And then I can say that X minus ... I'm trying to find a relationship between these two. And I know that ... X minus 15 plus 50 is going to equal 2X. So, 35 equals one X. So that would indicate that the facade on the hotel is 35 feet. On the office building is 70 feet.*

The solver continued to develop her solution activity while solving Tasks 5-9.

## Discussion

Abductions as Motivating Orientations for Future Activity. As Marie elaborated and extrapolated her hypotheses, her reflective scope widened in the sense that she could 'see' among many options for action and determine those which would help her make progress towards solving her particular problem. In this way, those future events *beckoned* to Marie for her to *actualize* specific trials, the results of which provided feedback to her evolving hypotheses. This reflective phenomena of formulating explanations in terms of future events is a critical aspect of abductive reasoning that involves the ability to coordinate images of action with one's evolving goals and purposes. The philosopher Bertrand de Jouvenel explained how images of action are in a sense projected and 'stored' into the future:

> *"Our actions seek to validate appealing images and invalidate repugnant images. But where do we store these images? For example, I "see myself" visiting China, yet I know I have never been there and am not in China now. There is not room for the image in the past or present, but there is room for it in the future. Time future is the domain able to receive as "possibles" those representations which elsewhere would be "false". And from the future in which we now place them, these possibles "beckon" to us to make them real." (de Jouvenel, 1967, p. 27)*

The role of anticipations. The Piagetian analysis of Marie explained her growth of awareness of her solution activity in terms of the idea of anticipation: "anticipation is nothing other than application of the scheme to a new situation before it actually happens" (Piaget, 1971, p. 195). As she solved variations of the original problem, Marie developed awareness of the structure of her solution activity, which in turn enabled her to anticipate results of potential solution activity prior to carrying it out with paper-and-pencil. Her anticipations existed and operated within the context of her scheme. Hence, relative to her scheme, Marie's anticipations brought forth results of applying the scheme.

By considering aspects of Marie's solution activity in terms of abductive reasoning, her anticipations can now be seen to be connected to her evolving hypotheses, and hence take on greater impact -- they were constituted within Marie's hypothesis-elaborating and hypothesis-testing activities, and thus helped her confer degrees of clarification and certainty in her on-going reasoning. In this way, problem solving for Marie was less about resolving problematic situations by revising her current scheme, but more about making her hypotheses work for her.

## References

Anderson, M. (1995, October). *Abduction.* Paper presented at the Mathematics Education Colloquium Series at the University of North Carolina at Charlotte, Charlotte, North Carolina.

Cifarelli, V. V. (1999). Abductive inference: Connections between problem posing and solving. In the *Proceedings of the 23rd Annual Conference for the International Group for the Psychology of Mathematics Education, 2,* 217-224. Haifa, Israel:Technion Printing Center..

de Jouvenel, B. (1967). *The art of conjecture.* New York, NY: Basic Books, Inc.

Fann, K. T. (1970). Peirce's theory of abduction. Martinus Nijhoff: the Hague.

Lobato, J. (1996). Transfer reconceived: How "sameness" is produced in mathematical activity. Unpublished doctoral dissertation, University of California, Berkeley.

Mason, J. (1995, March). *Abduction at the heart of mathematical being.* Paper presented in honor of David Tall at the Centre for Math Education of the Open University, Milton Keynes, UK.

Piaget, J. (1971). *Biology and knowledge.* Chicago: University of Chicago Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Yackel, E. (1984). Characteristics of problem representation indicative of understanding in mathematical problem solving. Unpublished doctoral dissertation, Purdue University.

# SOLVING EQUATIONS AND INEQUATIONS: OPERATIONAL INVARIANTS AND METHODS CONSTRUCTED BY STUDENTS.

**Anibal CORTES, Nathalie PFAFF**
CNRS-Université Paris 8, ESA 7021, Cognition et activités finalisées. 2 rue de la liberté, 93526, St. Denis France. (acortes@univ-paris 8.fr)

*Our study concerns mathematical justifications of elementary algebraic transformations provided by technical high school students (mean age: 17 years). We placed students in three categories based on their mathematical knowledge. Very few students were capable of explaining what is conserved in the transformation process.* ***The major principles for the construction of a teaching process are provided.***

Research concerning the mastery of literal elementary calculations has an undeniable social importance, as literal calculations constitute the bases necessary for dealing with the majority of mathematics chapters. However, despite its apparent simplicity, this type of calculation is the cause of a good deal of scholastic failure in mathematics. Thus, in a technical high school, the mathematics teachers are often confronted with students who fail at solving the most elementary algebraic exercises. For example, the analysis of a test on the solving of equations at the beginning of the year in two 10th grade technical classes (45 students, mean age: 17 years) showed that there was a real difficulty in solving equations. Here are a few examples:

| | *failure* |
|---|---|
| $4y + 49 = 93$ | 28% |
| $25 - 9y = 124$ | 57% |
| $(-4)(3-6x)+30=x(3-3,5+2,5)+100$ | 85% |

Most of our students considered the solving of equations as the application of an algorithm without any mathematical justification; they were therefore incapable of studying other mathematics chapters. Furthermore, our students worked in the best possible conditions: there were 23 students to a class, they worked 1/3 of the time in half-classes; they worked in "peer interaction" groups when solving problems; there were no discipline problems, etc. Failure in mathematics did not stem from the organization of the schoolwork.

We consider that the study of students' cognitive processes in learning is a central issue for research in mathematics education. Thus, from *a cognitive analysis of mathematical invariant properties involved in algebraic calculation* (section I) and *the identification of mathematical justifications of transformations that students constructed* (section II), we constructed a teaching method (section III) that allowed all class members to make considerable progress.

***Experimental work:*** Our research is based on different types of data collected during a two year period with 10th grade technical high school students.

a) Tests given to several classes requiring the solving of equations and inequations. We observed that students with very different levels of knowledge could correctly solve the same exercise with identical written solving steps. In general, the analysis of tests, when the answers were correct, did not provide sufficient information about the mathematical knowledge used by students.

b) The recording of student peer interactions in solving problems provided information about the difficulties that they confronted.

c) Individual interviews concerning the solving of equations and inequations. In particular, ten recorded interviews with students of different levels of knowledge allowed us to identify students' mathematical justifications of algebraic transformations.

***Theoretical framework:***

Several authors have analysed cognitive processes involved in algebraic resolution of problems, among them: Kieran C.(1994), Sfard A. and Lenchvski L. (1994), Matz M.(1982), Fichbein E. (1987), Tall D. (1995), etc. A complete analysis of these studies is impossible within the framework of this research. In agreement with Lerman S.(1998) concerning the development of an appropriate psychology for mathematics education, the present research provides a contribution.

On one hand, we aim to identify invariant mathematical properties involved in literal calculation. On the other hand, we want to identify the invariant properties used by students in order to undertake transformations. The concept of operational invariant is perfectly relevant in this type of analysis.

The concept of *operational invariant* was proposed by Piaget and later reformulated by G. Vergnaud who took into account the conceptual aspects of thought in learning processes. We call "operational invariant" specific knowledge which is often implicit. This knowledge provides a means to seek out pertinent information, enabling the subject to make inferences and anticipate situations. Thus, the organization of a subject's behavior, conditioning his performance, mainly depends on operational invariants that he has constructed in the classroom, as well as through his own personal work.

Vergnaud (1990) identifies two types of operational invariants: *concepts-in-action* (concepts pragmatically constructed by subjects) and *theorems-in-action* (for example, mathematical properties, true or false, pragmatically constructed by students). There are also more general invariants taken from another level of conceptualization. For example, Piaget (1950) considers conservation principles in physics (conservation of the energy, for example) as invariants of nature and as operational invariants of the physical thought: *"conservation principles constituting both absolutes of the considered reality and the operational invariants of deductive processes serve to analyze this reality"*.

In Cortés A. (1993) and in Cortés A. & Kavafian N. (1999), based on the analysis of errors made in solving equations, invariant mathematical properties (whose non-respect is the cause of most conceptual errors) are identified: the conservation of the equality and the priority of operations according to the undertaken transformations. The conceptualization of these properties allows the mental construction of three major operational invariants (principles): *the conservation of the equality, the respect of the priority of operations and the respect of different control types inherent in algebraic calculations*. These three principles guide the organizational behavior of experts in solving equations.

The present work concerns mathematical justifications which students provided for algebraic transformations. These justifications mainly concerned the conservation of equality and inequality.

## I - Mathematical properties conserved in algebraic calculation

In general, eighth grade school textbooks summarize the method of solving equations by the following two rules:

*1) One adds (or subtracts) the same number to each side of the equation obtaining a new equation*

*2) One multiplies (or divides) by the same number each side of the equation obtaining a new equation*

Firstly these rules make explicit mathematical properties considered as self-justified or evident which become operational invariants for 8th-grade students at the beginning of the learning process.

Furthermore, these rules have an operative purpose; they are tools to produce transformations. In this sense, these two rules contain the most important aspects of the algebraic method. On one hand, they explain that algebraic transformations are made **on** the equation, and on the other hand, they affirm that a sequence of equivalent equations that conserve the solution(s) is obtained. These rules also concern the algebraic transformations of unknown terms, particularly the addition or subtraction of an unknown number. During the learning process, students have to articulate these two rules with the other algebraic transformations that are made on only one side of the equation (development, factorization, reduction of terms) seen previously.

***Allowed transformations conserve the equality*****;** the "equal" sign is conserved: another equality is obtained. The conservation of the "equal" sign is an **observable** phenomenon in different mathematical processes: the transformation of an equation, a system of equations, a function, etc. In each case the "equal" sign conveys a precise signification which is conserved. Transformations also conserve inequalities.

*What is conserved in the transformation of equations, inequations, etc.?*
For example, the equation: $(-8)3x+15 = 2x-50$ is an equality between two expressions containing the unknown x. This equality can be true for a particular value of x (then the two expressions are equivalent) or false in other cases. An allowed transformation, e.g.: $(-8)3x+15\mathbf{-15} = 2x-50\mathbf{-15}$ provides a means *to conserve the truth-value present in the first equation*: possible equivalencies (and solutions) are conserved; as are false equivalencies. In the case of the re-writing of a function, conservation of equality expresses the conservation of an identity. In the solving of inequations, transformations also conserve the truth-value.

***The conservation of the truth-value,*** common to all transformations, is a principle that expresses the mathematical filiation of all algebraic transformations. For the expert (the teacher, for example), *this principle is an operational invariant*, which is often implicit. From a cognitive point of view, this principle is the major property of literal calculation.

The conservation of the solution(s) is a very important property. However, it can only be verified once the solution has been calculated (at least within the framework of the school). Therefore, the conservation of the solution, even if it is announced, cannot become an operational invariant that can guide thought in the solving process.

## II) Students' mathematical justifications of algebraic transformations which are made identically on both sides of the equation.

Students of 10th-grade classes do not use the self-justified rules previously seen. They use economic rules, for example: "*A term passes to the other side by changing sign*"; "*the coefficient of the unknown passes to the other side by dividing*", which lead to more abbreviated notations. Individual interviews allowed us to identify the meaning that students gave to the rules that they used in solving equations, inequations, etc. We could thus distinguish three categories of students according to the mathematical knowledge they used:

*a) Students that justified transformations as being identical operations on the two sides of the equation.*

*b) Students that provided arithmetical justifications for transformations used in processing numbers.*

*c) Students that did not provide any mathematical justification for transformations.*

**II-a) Students that justified algebraic transformations** ***as being identical operations carried out on both sides of the equation.***

Only very few students (1 or 2 per class) provided this justification for the transformations which *"transfer terms to the other side..."*. They thus reproduced the solving method taught in 8th-grade classes. Their operational invariant was therefore a mathematical property considered as evident, which was linked (implicitly or explicitly) to the conservation of the equality.

These students easily justified multiplicative transformations as being operations (division or multiplication) identically performed on each side of the equation. For example, in solving the equation 3x+15 = -50, preliminary processing of the coefficient (requested by the experimenter) was correctly performed: x+15/3 = -50/3. The provided justification was *"if you don't divide everything by 3, it's no longer equal"*, also implicitly expresses the conservation of the equality. *We observed that the constructed operational invariant allowed students to check the validity of unusual transformations.* In a similar manner, the change of sign (-24x=-65 => 24x=65) was justified as being the multiplication by (-1) of the two sides of the equation.

Some students knew the justification for multiplicative transformations but did not know the justification for additive transformations. However, they could distinguish between the two types of transformations. For example Brice, in solving the equation: -24x+15=-50, was not able to explain why the intermediate equation -24x=-50-15 was correct. The rule used (*"a term passes to the other side by changing sign")* expresses a non-evident mathematical property without justification; this property is an operational invariant of the type *theorem-in-action*.

It is interesting to notice that these students could very rapidly construct this justification once they were aware:

-That the *equation is an equivalence* between two expressions for a particular value of the unknown.

-That the filiation between multiplicative transformations (which they could justify) and additive transformations consists *"in performing the same operation on both sides of the equation"*.

Thus, Brice justified *"the transfer of 15 by changing sign"*, by replying that"... *it is a mathematical property; if I remove something here, I have to remove the same thing there"* and writing: -24x+15-**15**=-50-**15**. It is very probable that this mathematical property was known at a given moment.

The experimenter then asked: *"what is conserved in this transformation process?"*. Brice replied, "equality". Notice that the word "conservation" was associated with the word "equality" by the most brilliant students. They did not associate with the word "conservation" the conservation of solutions. Only one student provided the most general definition of the conservation of the equality: *"you conserve the truth value"*. This principle, a fundamental operational invariant of the process, was implicit or rather difficult to express for students.

**II-b) Students that provided arithmetical justifications for transformations concerning the transfer of numbers.**

Approximately half of the class justified *"the transferring of numbers to the other side of the equation"* by means of arithmetical knowledge. For example, in the case of additive transformations students made the following justifications: *"subtracting becomes adding"* or vice versa. This solving model, based on the recognition of the operation and its opposite operation, is the first model taught in 6th-grade classes for solving the simplest equations (x + a = b or ax = b). *The arithmetical justification* is an operational invariant suited to checking the transfer of numbers but it is difficult to apply to additive transformations of terms containing the unknown. In this case, students used an effective rule *("the term is transferred to the other side by changing sign")* but without any justification.

The transferring of the coefficient of the unknown appeared at the end of the solving process when the student was confronted with an equation of the type ax = b. The undertaken transformation was justified as follows: *"the coefficient is transferred to the other side by dividing"* or, more precisely by means of proportionality. For example, in solving the equation 24x = 65 : *"if you want to know the value of only one x, you divide 65 by 24"*. This *self-justified mathematical property* constitutes an operational invariant providing a means to decide, in a reliable manner, what the resulting quotient of the transformation is, even when the coefficient is a negative number. Arithmetic only allows students to justify the solving of equations of the type ax + b = c. Applying the arithmetic method to more difficult situations led students to fail or to use rules without mathematical justification.

***Failure in adapting the arithmetical model.*** Here are some examples:

*-The transfer of terms containing the unknown*: some students constructed false rules. For example, two students correctly solved the equation 4y+49=93 by referring to the operation and its contrary. These same students, however, failed later on while solving the equation 13x+28 =5x+49. Transferring the term 5x to the other side of the equation followed an erroneous rule: 13x+**5x**=49-28 (the term 5x did not change sign). Checking the validity of this transformation was no longer possible.

*-The processing of the unknown coefficient*: for example, in solving the equation 3x+15=-50, the preliminary processing of the unknown coefficient (asked during the interview session by the experimenter) led to error. The student, using the arithmetical procedure, wrote the erroneous equation: x +15=-50/3. Checking the validity of this transformation was no longer possible.

*-The multiplication of the two sides of the equation by (-1).* This transformation can not be justified using arithmetic. We observed that our students considered this fundamental transformation as the transferring of the minus sign to the other side of the equation. Thus, our students used rules without any justification, such as: *"the minus sign is transferred to the other side as a plus."*, *"I remove the minus sign from the two sides of the equation"*, etc. This led to multiple errors. However, this type of transformation is necessary in solving systems of equations and inequations.

***Adaptation of arithmetical methods*:** Nearly half of our students justified the transfer of the unknown coefficient by arithmetic. Transformations are made applying the rule: *"...it is transferred to the other side of the equation by dividing."* Additive transformations were undertaken, for the most part, by following the rule without justification: *" you transfer a term on the other side of the equation by changing the sign."* The mental construction of the justification of the transformation concerning the coefficient allowed these students to differentiate the two types of transformations.

This differentiation of transformations provides a means to construct a method effective for processing a good number of the proposed exercises. The two rules cited previously are applied as an algorithm: one first undertakes additive transformations and when an ax=b type of equation is found, a multiplicative transformation is then undertaken. These students could thus solve equations of the type ax+b=cx+d and some inequations.

### II-c) Students that did not provide any mathematical justification for algebraic transformations.

At least half of our students used transformation rules without any mathematical justification. Take, for example, a conversation between Mathieu, a student of the class, and the teacher (Mrs Pfaff):

**Mathieu:** $2x - 4.2 = -5.4$; that makes $2x = -5.4 + 4.2$

**Teacher:** why "plus" 4.2

**Mathieu:** You transpose it and the "minus" sign is transferred as a "plus" sign. Therefore,... in any case, I learned it like that. I do it automatically. In my head... that, for me, doesn't really mean anything.

**Teacher:** For you, writing "plus" 4.2 doesn't mean anything?

**Mathieu:** Here, I do it directly and **I don't try to understand**. I put the "plus" sign, and that's how I transfer the term.

**Teacher:** In your opinion, did you learn it like that?

**Mathieu:** In my opinion, no, I didn't learn it like that. But, for me, I've forgotten it (the justification), I do it automatically...

Mathieu summarizes effectively the behavior of many students: they use a rule without knowing its justification and in addition, what seems more serious: they do not feel the need to know this justification. Nevertheless, Mathieu differentiated additive transformations from multiplicative transformations. This implicit differentiation of transformations allowed him to construct a solving method that resembles a relatively efficient algorithm. **Algorithm of resolution**: he first processed additive transformations according to the rule *"transfer to the other side by changing signs"* and, when he had an $ax = b$ type of equation, he then applied a multiplicative transformation: *"the coefficient transfers by dividing"*. His operational invariants were not evident mathematical properties (*theorems-in-action*), without justification.

Many students, that were not able to justify transformations they use, did not construct an efficient algorithm. Here are some examples of false rules pragmatically constructed by students:

- *Only one rule*: Cyrille's method was limited to the application of one rule (*"you transfer to the other side by changing signs"*) to process additive and multiplicative situations. For example, the equation $4y + 49 = 93$ became $4y = 93 - 49$, then he made the following error: $y = 93 - 49 \mathbf{- 4}$.

- *Quotient inversion*: For 1 or 2 students per class the equation $4y = 44$ yielded $y = 4/44$ instead of $y = 44/4$. These students knew that the ensuing transformation leads to a quotient, but they did not have a reliable rule to decide which quotient was correct.

-*Erroneous change of sign in a division*: For 2 or 3 students per class the equation: $\mathbf{-9}y = 99$ became $y = 99/\mathbf{9}$. For these students this error was systematic and it corresponded to a false rule that they had constructed: *"in the transferring of a coefficient by dividing, the sign changes"*, as in additive transformations. Thus, properties of additive transformations **slip towards multiplicative transformations**. Students who had great difficulty in algebra constructed these types of rules confusing the two types of transformations.

### III - How to teach in heterogeneous classes?

We were confronted with the problem of constructing a teaching process for algebraic calculations that could aid weak students, whilst at the same time allowing all students to progress. Thus, instead of revising the solving of equations, we asked our students to solve inequations, which the entire class failed at, notably in using transformations implying the change of direction of the unequal sign.

*The main points of our teaching method were as follows:*

a) *Concept of inequation*: inequations were constructed as relationships between linear functions that our students were familiar with. For example:

$g(x) = -1.2x+1$ and $f(x) = 3.4x \Rightarrow -1.2x+1 > 3.4x$.

b) *Concept of solution and concept of truth-value*: students had to calculate (by substitution in given inequations) whether given numbers (5; 3.2…) were solutions or not.

c) *The set of solutions* was graphically determined without problems.

d) *The algebraic solving of inequations* was proposed in order to verify the set of solutions graphically determined: *failure*. This failure allowed the teacher to detail algebraic transformations, which were presented as being identical operations on the two sides of the inequation. We guided students to analyze which transformations conserved the direction of the unequal sign and which transformations did not conserve the direction of the unequal sign.

The change of direction of the unequal sign was introduced as including, on one hand, the multiplication by (-1) of the two sides of the inequation (the change of sign) and on the other hand, the division or multiplication by a positive number.

This transformation was justified by means of a numerical analogy. For example: $40 > -20$ became $(-1)\,40 < (-1)\,(-20)$ and then $-40 < 20$ by observing that the change of direction allows us to conserve the truth-value of inequality. The use of a numerical analogy provided students with a means to justify this transformation which had always seemed mysterious and also provided a tool for checking this type of transformation.

e) *Conservation of the inequality*: students were required to complete sequences of inequations (in which transformations were explicitly noted on the two members) using the symbols "<" or ">". For example:

$-3x+4 > 2 \Rightarrow -3x+4-4 \,\ldots\ldots\, 2-4 \Rightarrow -3x \,\ldots\ldots\, -2 \Rightarrow -3x/(-3) \,\ldots\ldots\, -2/(-3) \Rightarrow x \,\ldots\, 2/3$

f) Resolution word problems and inequations.

During the learning process, notably during the correction of the exercises, the teacher (Mrs. Pfaff) noted transformations on the two sides of the inequations and reminded students of the conservation of the inequality and the truth-value (including the conservation of solutions). We think that the constant reminder of these conservations allowed students to fix them in semantic memory. These conservations serve to validate unusual and problematic transformations.

At the end of the teaching cycle, we observed considerable progress among all class members in solving *equations and inequations*. In the following table we reported the number of students failed on solving equations and inequations at the pre-test and at the post-test (the total number of students of the class is 23). Here are a few examples:

| **Failure** | **Pre-test** | **Post-test** |
|---|---|---|
| $4y+49=93$ | 6 | 1 |
| $(-4)(3-6x) + 30 = x(3-3,5+2,5) + 100$ | 20 | 1 |
| $11x<8x+7$ | 21 | 2 |
| $250-90y>124$ | 20 | 3 |

Only three students made conceptual errors, at the post-test, two months after end of the experiment.

**CONCLUSION:** The present work seems useful to describe other groups of students given that the identified phenomena are present, although most likely to a lesser degree, in others classes in other types of high schools. We have identified different types of operational invariants and methods constructed by students; the main characteristic that differentiated students was their level of knowledge:

a) Very few students justified transformations by means of *self-justified mathematical properties* (considered as evident), which concern (implicitly in most cases) the conservation of the equality and the inequality. These operational invariants can be applied to the solving of inequations, systems of equations, etc.

b) Approximately half of the class justified the transfer of numbers by arithmetic. These *self-justified mathematical properties*, operational invariants, can not be applied to the solving of inequations, system of equations, etc. Their solving methods function much like an algorithm.

c) Approximately half of the class used transformation rules (sometimes false rules) which express non evident mathematical properties without any justification; these operational invariants are *theorems-in-action*. The checking of unusual transformations was no longer possible. Students' method for solving equations was an algorithm which could not be applied to other algebraic calculations.

On one hand, the same transformations are used in the solving of equations, inequations, systems of equation, etc. These transformations conserve equalities and inequalities (observable in the written sequences) : the truth-value of mathematical expressions is conserved.

On the other hand, students' solving methods depended, in most cases, on a juxtaposition of arbitrary rules without mathematical justification; the checking of the validity of transformations was no longer possible. Furthermore, students would not change their " economics rules " like *"a term pass to the other side of the equation changing sign"*...

Thus, it was necessary to guide students in constructing mathematical justifications for "economic" transformation rules, which would allow them to construct checking processes. Constructed justifications provide a means for students to confront and succeed in algebraic calculations. This study hopes to provide a tool for teachers in preparing teaching courses.

**Cooper TJ, Boulton-Lewis GM.,Atweh, Pillay, Wilss,and Mutch (1997)** The transition from arithmetic to algebra: initial understanding of equals, operations and variables. *Proceedings of the 21st PME conference.* Vol 2,89-96. Lahti, Finland

**Cortés A.(1993)**; Analysis of errors and a cognitive model in the solving of the equations. *Proceedings of the seventeenth PME Conference. Japan, Vol. I; 146-153*

**Cortés A. (1998)** Implicit cognitive Work in putting word problems into equations form. *Proceedings of the 22nd PME conference.* Vol. I, 66-81. Stellenbosch. South Africa.

**Cortés A. ; Kavafian N. (1999)** Les principes qui guident la pensée dans la résolution des équations. *Revue "Petit x". Editeur : Irem de Grenoble. No51 ; 47-74.*

**Fichbein Efraim (1987)** Intuition in Science and Mathematics. D. Reidel Publishing Company Ed. Dordrecht, Holland.

**Kieran C.(1994)** A functional approach to the introduction of algebra.*Proceedings of the 18th PME conference.* Vol. 1, 157-175. Lisbon, Portugal.

**Lerman S.(1998)** A moment in the zoom of a lens: towards a discursive psychology of mathematics teaching and learning. *Proceedings of the 22nd PME conference.* Alwyn O. & Newstead K. Ed. Vol. I, 66-81. Stellenbosch. South Africa.

**Matz M.(1982)**. A process model for high school algebra errors. In Sleeman & Brown(Eds), *Intelligent tutoring systems*, London: Academic.

**Payne S., & Squibb H. R**. (1990). Algebra Mal-rules and cognitive account of error. *Cognitive science*, 14,445-481.

**Piaget J. (1950)** Introduction à l'épistémologie génétique - La pensée physique. PUF

**Tall D. (1995)** Cognitive growth in elementary and advanced mathematical thinking. *Proceedings of the 19th PME conference.* Vol. 1, 61-75. Recife, Brazil.

**Sfard A.& Linchevski L.(1994)** The gain and the petfalls of reification. The case of algebra. *Educational studies en Mathematics*, 26,191-228

**Vergnaud, G** (1990), La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques.* Vol.10 N°23, 133-170.

# The Flow of Thought Across the Zone of Proximal Development between Elementary Algebra and Intermediate English as a Second Language

Bronisuave Czarnocha
Eugenio Maria de Hostos Community College, CUNY, New York City
Vrunda Prabhu
William Woods University, Missouri

## Abstract

*A teaching experiment in correlating the instruction of courses in Elementary Algebra and Intermediate ESL is described whose results suggest a measurable transfer of thought organization from algebraic thinking into written natural English. It is shown that a proper context to situate this new effect is the Zone of Proximal Development of L. Vygotsky.*

## Introduction

The relationship between Mathematics and Language teaching has been the topic of many papers and presentations [1], [2], [4],[9]. Yet the literature and research on the subject suffer from several shortcomings. First, the majority of the research deals with the role of language in learning mathematics, leaving the reciprocal relationship, that of the influence of learning mathematics on the development of language, almost totally unexplored. Second, although several benefits of writing as an instructional tool in teaching mathematics have been proposed - such as a better understanding of conceptual relationships [2] or the facilitation of "personal ownership" of knowledge [4], [10], there has been, till recently, little evidence to explicitly demonstrate these benefits [11]. Finally, there is a relative absence of theoretical considerations that could provide a context in which to properly situate the reciprocal relationship between the development of mathematical understanding and the mastery of language.

ESL literature presents us with more or less the same situation and focuses on the role of the mathematics teacher as " a teacher of the language needed to learn mathematical concepts and skills." [5]. The methodology of classroom practice based on this principle was formulated in [6]. An important theoretical distinction in the area of second language acquisition has been introduced by Cummins ([5]), who asserted that the process of language acquisition has at least 2 distinct levels: the Basic Interpersonal Language Competency (BILC) level of everyday use, and the Cognitive Academic Language Proficiency (CALP) level.

This presentation addresses the shortcomings listed above. A brief discussion of certain ideas of Vygotsky in [13] outlines a context in which the relationship between mathematics and language can be situated. This is followed by a new and interesting result obtained during a teaching experiment at CUNY's Hostos Community College, in which an Elementary Algebra course was pedagogically linked with a course in English as a Second Language (ESL), most probably demonstrating an influence of mathematical thinking on the development of descriptive writing.

## Theoretical Background

The literature contains sporadic hints about the relationship between mathematical understanding and the acquisition of language. To the degree that writing is like problem solving, as Kenyon [7] claims, one might think about this relationship as determined by a common set of problem-solving strategies. This point of view, which doesn't take into account the peculiarities of each of the disciplines, is strongly supported by Anderson's Adaptive Control of Thought theory [1].

A point of view that gives justice to the richness of relationships between thought and language might perhaps be found in (early) Vygotsky ([13]). There, thought and language are seen as being in a "reciprocal relationship of development." [8]. "Communication presupposes generalization..., and generalization... becomes possible in the course of communication" ( p.7, [13]) – in other words, in order to communicate we need to think; and in order to think, we need to communicate. Such a view opens, in a very natural way, the possibility that thought - in our case, mathematical thought - might be able to shape natural language. One of the ways through which this process can take place is across the Zone of Proximal Development (ZPD). (Ch.6, [13]).

The ZPD arises in Vygotsky's thought through his distinction between spontaneous and scientific concepts. It represents the depth to which an individual student can develop, with expert help, his spontaneous concepts concerning a particular problem - as opposed to his ability to do it alone.

Valsiner had noted that the development of the ZDP can also be furthered if the environment is structured in a way that leads the student to use elements which are new to him, but reachable from his ZDP [14]. One of the essential characteristics of the upper level of ZDP, as compared to the level of the corresponding spontaneous concepts, is its higher degree of structural systematicity. In our experiment, the abstract character of elementary algebra has created exactly that type of ZPD with respect to the "spontaneous" level of natural English.

## Experimental Realization

To confirm Vygotsky's highly dialectical view one would clearly need to detect the presence of two directions of developmental progression: the acquisition of English under the influence of mathematical thinking and the acquisition of mathematical understanding under the influence of the use of English. While the first direction is the main topic of this presentation, let us mention that the existence of the second has been confirmed, for the first time, in a recent experiment by Wahlberg [13]. Measuring the increase in the students' understanding induced by essay-writing; she observed a substantial (80%) increase in the experimental group as compared to the control group.

**Elementary Algebra/Intermediate ESL teaching experiment.***

The general goal of the ESL sequence at Hostos is to develop what Cummins ([5]) calls the Cognitive Academic Language Proficiency, and what Vygostky calls the language of "scientific concepts". Our experiment had two goals: to see how far Algebra can help in that process, and to investigate the cognitive relations in the acquisition of both.

**Methodology**

A group of seventeen students was enrolled in a class of Intermediate ESL and in a remedial class of Elementary Algebra taught in English. In the previous semester these students passed 2nd low level ESL as well as Basic Arithmetic, the first remedial mathematics course. The Algebra class was the only class they were taking in English, and thus constituted their only exposure to academic English and possibly to English in general. Although the classes were separate, the communication between the instructors was very tight, involving weekly meetings, exchanging materials, and visiting each other's classes. The methodology of the experiment was based on two assumptions. First, since we were interested in the influence of the algebraic language upon the natural one, we needed to verbalize the algebraic language to a maximum level possible. That meant we needed to make the symbolic notation of algebra explicit in speech and/or writing - to verbalize the procedural steps and the content of algebraic thinking. Second, these elements, having been made explicit in their algebraic context, needed to be transferred into the context of the ESL class, both on the semantic and the grammatical level.

As a result, student discussions in the Algebra class often involved a level of academic discourse somewhat above the students' capacity at the given time.

This effort the students had to make to communicate the comparatively abstract mathematical ideas in English is, we think, is at the root of their very particular linguistic improvement. At the same time, the ESL class deliberately involved discussions of the linguistic peculiarities of algebraic language, such as the role of word order and sentence structure. Below are examples of specific instructional strategies in both classes.

New algebra instructional strategies.

1. Verbalization of algebraic procedures.

Example: Solving linear equations: $2X + 5 = 9$

Solution Steps (to be written by students)

$2X + 5 - 5 = 9 - 5$ *First, I add -5 to both sides of the equations in order to eliminate +5*

---

-------* The description of the experiment and of its results is based on The Final Report of the ESL/Elementary Algebra Teaching Experiment supported by the New Visions Program Grant of CUNY, July 1998 – M. Pujol, B.Czarnocha *on the left side.*

$2X = 4$ *Second, I cancel the opposite numbers and add the like terms*

$2X/2 = 4/2$ *Third, I divide both sides by 2 in order to have X alone*

$X = 2$ *The answer is $X = 2$.*

2. Explication of algebraic symbolism through writing paragraphs

a) *Write a paragraph explaining the difference between $3*5$ and $5*3$. What does it mean to you that $5*3 = 3*5$?*

b) *What is the difference in the meaning of the equality sign in $3*16 = 48$ and in $X + 5 = 12$?*

3. Analysis of algebraic rules and principles.

a) *Compare the rule for the addition of signed numbers with different signs with the rule for the multiplication of signed numbers with different signs.*

c) *Write an explanation to Jose who missed a couple of classes, about how to do the problem below. Explain to him the order of steps in the procedure, warn him against the possible errors he might make, remind him of the rules which justify your the steps of the solution.* $-[2(3X - 5Y) - 3(Y - X)] - 4(2X + 3Y) =$

New ESL instructional strategies.

The goal of the-math related ESL exercises was to extend the meaning and application of algebraic words and concepts into natural English.

a) Expressing the same Math concept in different ways in English

Example. **Solving Linear Equations**

Instruction: Fill in the blanks with the appropriate word.

$$x + 16 = 20$$

- *Sixteen .....**added**.........to an **...unknown ...** number .....**is**.. twenty.*
- *If an **...unknown ...**number is ...**increased**.. by sixteen, the ...**result**.... is twenty.*
- *The ....**sum**.. of an **.unknown** . number ..**and**.. sixteen ...**equals**.. twenty.*
- *Sixteen ...**more**... than an **...unknown ...**. number is ...**equal**.. to twenty.*

These exercises allowed students to understand how a specific Math idea or concept could be phrased in many different ways in English. This provoked opportunities for thinking and internalizing the basic English sentence structure.

b) Word order exercises.

Instruction: ***Put the following words and expressions into the right order:***

*1- temperature - by +10 - the - by - the - decreases - evening - degrees*

*2- multiply - he - the number - needs to- in the parentheses- by- the numbers -*

*3- much - how - you- your sister - to - money- owe- do?*

*4- makes - of - John - same- as - do - money - every month- the - I -amount*

*5- If - perform- the - you - you'll- the number 48 - multiplication- get-*

Serious thinking and discussion about what was said and how it was phrased accompanied these exercises. Students learned that the way concepts and numbers are put together in the algebraic language is essential for understanding algebraic operations. By paying attention to the word order in algebra, students became sensitive to word order in English.

c) Editing exercises.

Instruction: Please correct the following paragraphs, written by different students, not only for correct ideas but for mistakes involving any of the grammar rules studied so far.

$(x^a)^b$ *– I think that raise a power to another, I need to put the variable and then multiply the exponent.*

$(xy)^a$ *– When you have two variable in the parenthesis and you raise to any power. You have to multiplied each variable with same power.*

*--I need to get each variable to the power separately. I need to multiply the variable with the raise power.*

*--It is when you have two variable raising to a power. I solve this raising separately each variable to the power. For example, I do this because I multiply the variable by the exponent.*

d) A long-term (6-week) essay, written on a word processor, with 3 drafts discussed with the instructor. The topic was In Between Two Cultures; the students were supposed to compare and contrast their life experience in the Dominican Republic and in New York City.

## Data collection and analysis.

As has been stated above, the teaching experiment had two goals: to use algebra to help in the development of natural English, and to investigate the possibility of a cognitive relationship between the acquisition of both. Vygotsky suggests such a possibility when he asserts (p.160,[13]): "one might say that the knowledge of the foreign language stands to that of the native one in the same way as knowledge of algebra stands to knowledge of arithmetic... There are serious grounds for believing that similar relations do exist between spontaneous and academic concepts".

For the purpose of the present discussion, the main tool of analysis were the long-term essays on the topic In Between Two Cultures which the students wrote in the course of the semester. The process of writing was important because "Written speech assumes much slower, repeated mediating analysis and synthesis, which makes it not only possible to develop the required thought, but even to revert to its earlier stages, thus transforming the sequential chain of connections in a simultaneous, self-reviewing structure. Written speech thus represents a new and powerful instrument of thought"[8].

To assess the changes in the students' written mastery of English, we first used the holistic assessment of the ESL instructor - a standard way of judging student essays in English courses. Next, we translated this judgment into syntactic components. Finally, we compared these with the corresponding components in the essays of a control group. We chose as our control group a past class of the same ESL instructor, which wrote an essay on the topic Our Family Conflicts. This topic was judged to be the closest in meaning to the topic In Between the Two Cultures, assigned to the experimental group.

The judgment of the ESL instructor after reading all the essays of the experimental group was that they were more cohesive. As cohesiveness is closely related to the use of connectors - words such as "because", "yet", "although", etc.- all the connectors used by all students in their essay were categorized, counted, averaged per 22-line page, and the results compared with the corresponding numbers from the control group. _Our conclusion was that there was an average 15% increase in the number of connectors and subordinating clauses in the essays of the experimental group. This confirmed the ESL instructor's assessment that the long-term essays of the experimental group were more cohesive than the essays of the students who did not participate in the instructional link under discussion.

**Comparison of the use of connectors in student writing:**

| | **Experiment** | **Control** | **% increase** |
|---|---|---|---|
| **Time** | 228/**5.16** | 217/**4.8** | **8%** |
| subordination (when) | 114/**2.58** | 106/**2.35** | **10%** |
| others | 38/**.86** | 53/**1.17** | **-36%** |
| connectors | 66/**1.49** | 56/**1.24** | **20%** |
| **Cause** | 181/**4.09** | 152/**3.37** | **21%** |
| subordination (because) | 140/**3.17** | 118/**2.62** | **21%** |
| others | 20/**.45** | 20/**.44** | **2%** |
| connectors | 21/**.48** | 14/**.31** | **55%** |
| **Purpose** | 29/**.66** | 20/**.44** | **50%** |
| subordination | 27/**.61** | 20/**.44** | **37%** |
| connectors | 2/ | | |
| **Condition** | 21/**.44** | 28/**.62** | **22%** |
| **Contrast** | 29/**.66** | 22/**.49** | **39%** |
| subordination | 14/**.32** | 18/**.40** | **20%** |
| connectors | 13/**.29** | 4/**.09** | **222%** |
| **Place** | 19/**.43** | 3/**.07** | |

| | | | |
|---|---|---|---|
| **Addition** | 46/**1.04** | 38/**.84** | **24%** |
| TOTAL | 553/**12.52** | 489/**10.86** | **15%** |

**Subordinating Conjunctions**:

**Time**: When (the most common one)/ As soon as/ Before/ After/ etc.

**Cause**: Because ( the most common one)/ As/ etc.

**Purpose**: In order that/ So that / etc.

**Condition**: If / Unless/ etc.

**Contrast**: Although/ Even though/etc.

**Place**: wherever/ etc.

**Transitional Words or Connectors**:

**Time**: First/ Second/ Finally/etc.

**Cause**: Therefore/ As a result/ In consequence/ Because of this/etc.

**Contrast**: However/ Nevertheless/ On the contrary/etc.

**Addition**: In addition/ Moreover/ Also etc.

**Example**: For example/ In fact/ etc.

These measurements provided an independent confirmation of the holistic assessment that the essays written under the influence of algebraic thinking were more cohesive, more thoughtfully written. Despite the novelty of this observation one should not be surprised by it. Algebra, as an abstract area, depends a lot on the relationship between different concepts, ideas and mental actions. Connectors and subordinating clauses are those particles of language which are used to express the relationship between ideas, events or facts; these are words such as "because", "in order to", "finally", "if....then." They are used to express cause and effect relationships, conditions, reasons, and contrast; thus, they seem to be closely related to what is called a critical (or analytical) mode of thinking. ". A correct use of connectors determines the organization of ideas within an essay. The increase in the (correct) use of these linguistic tools meant that there was an increase in the number of relationships between ideas, their better organization expressed by our students in their writing, making it more cohesive. The correlation of the ESL syllabus with the Algebra course - whose dense mathematical relationships, when translated into natural language with the help of connectors, were able to penetrate the simpler language of descriptive writing - induced an increase in the level of thinking effected by the ZPD.

Bibliography

[1] Anderson, J.A. (1982) *Acquisition of cognitive skill* San Francisco, Freeman

[2] Birken, Marcia (1989) Using writing to assist learning in college mathematics classes. In P. Connoly and T. Villardi (Eds) *Writing to Learn mathematics and Science*. New York, NY: Teachers College Press,

[3] Cuevas, G. (1984) Mathematics Learning in English as a Second Language, *Journal for Research in Mathematics Education* V 15(2) 134

[4] Connoly, Paul (1989) Writing and ecology for learning. . In P. Connoly and T. Villardi (Eds) *Writing to Learn mathematics and Science*. New York, NY: Teachers College Press,

[5] Cummins, J. (1980) The entry and exit fallacy in bilingual education. *NABE:Journal for the Association for Bilingual Education*. 4(3), 25-60.

[6] Dale, T. C. & Cuevas, G. J. (1987 Integrating Language and Mathematics Learning. In J. Crandall (Ed*) ESL through Content Area Instruction,* Englewood Cliffs, NJ Prentice Hall Regents.

[7] Kenyon, R.W. (1989) Writing *Is* Problem Solving. . In P. Connoly and T. Villardi (Eds) *Writing to Learn mathematics and Science*. New York, NY: Teachers College Press,

[8] Kozulin, A. Vygotsky in Context. In Vygotsky, L. S. *Thought and Language*. The MIT Press, Cambridge, Massachussets.

[9] Luria, A.R., Yudovich, F. (1971*) Speech and the development of mental processes in the child*. Baltimore, MD: Pinguin

[10] Mett, Coreen (1998) Writing in mathematics: evidence of learning through writing. *The Clearing House*, 62, 293-296

[11]Powell, A. B. Lopez, J. (1989) Writing as a vehicle to learn mathematics: A case study in. Connaly, R., Vilardi, T. (eds) *Writing to learn Mathematics and Science*

[12]Wahlberg, M. (1998) The effects of writing assignments on second-semester calculus students' understanding of the limit concept. Presentation, *3rd RUMEC International Conference in Advanced Mathematical Thinking.*

[13]Vygotsky, L. S. (1986) *Thought and Language*. The MIT Press Cambridge, Massachussets

[14]Valsiner (1999) The Encoding of Distance: The Concept of the Zone of Proximal Development and its ingredients. In Lloyd, P., Fernyhough,C. (eds.) *Lev Vygotskyi: Critical Assesments. NY Routledge*

# A DIDACTIC SEQUENCE FOR THE INTRODUCTION OF ALGEBRAIC ACTIVITY IN EARLY ELEMENTARY SCHOOL

**Jorge Tarcísio da Rocha Falcão, Anna Paula Brito Lima, Cláudia Roberta de Araújo, Mônica Maria Lins Lessa, Mônica Oliveira Osório.**

*This research is aimed to propose and evaluate a didactic sequence for the introduction of algebraic activity in elementary school (2nd grade). This sequence was composed by five interconnected modules, addressing aspects considered central in the conceptual field of algebra, like identification and modeling of transformations, symbolic representation, manipulation of relations of difference and equality, and manipulation of unknown quantities symbolically represented. Results show the possibility of introducing this kind of mathematical activity since very early, in parallel with arithmetics, provided that the passage from the manipulation of known to unknown quantities is firstly based on natural language and preceded by preparatory activities, and the literal representation of unknowns carefully assisted.*

## 1. Introduction

The program of elementary school mathematics in Brazil, as in many other countries, proposes that algebra must *wait* until 6th or 7th grades in order to be introduced, since arithmetics "precedes" conceptually algebra. This phenomenon of didactic transposition of algebra is also supported by considerations of psychological order, like those proposing a "concrete" character of arithmetics, and a "formal-abstract" character of algebra (Piaget & Garcia, 1983), what justifies, in terms of developmental psychology, the precedence of arithmetics over algebra in school programs. Recent research data shows, however, that the pure arithmetics "immersion" during the first years of elementary school can create a didactic obstacle to the introduction to algebra, specially in terms of the manipulation of unknown quantities in problem-solving activities (Brito Lima, 1996; Lins Lessa, 1996; Brito Lima & Da Rocha Falcão, 1997). In the other hand, arithmetics and algebra are effectively different mathematical activities, what can be discussed in psychological (Cortes, Vergnaud & Kavafian, 1990), historical (Rashed, 1984) and mathematical (Bodanskii, 1991) terms. There are of course conceptual connections between these two domains, but a necessary temporal order of presentation in terms of school curricula must be submitted to criticism. In this research, we propose an organized set of activities, in the context of a didactic sequence, in order to offer to students of elementary school level the opportunity to start algebraic activity in parallel with other mathematics subjects. The main goal of this proposition was verifying clinically how the group of students who took part in the research received these activities.

## 2. Procedure

The sample of this study was constituted by 23 students (12 boys, 11 girls) of the 2nd grade (elementary level) of a public school from Recife (Brazil), with ages varying from 10 to 12 (modal age 11). Five teachers-researchers and two research-assistants (undergraduate students) formed the research team, and two teachers and two research-assistants (one of them charged of videotape record) conducted each work session. The didactic sequence covered a period of seven months, during the school year of 1999, with two weekly sessions of one hour each. The research contract mentioned clearly that we would work on mathematics during the year, the work being registered in videotape for later study, and that they would work individually, in small groups and in the context of the whole group. The entire didactic sequence was composed by four interconnected modules, each of it addressing an aspect considered relevant in the conceptual field of algebra: **Module 1** explored the activity of describing processes and transformations in general terms, exploring ideas connected to the concept of function; **Module 2** explored further one aspect initially proposed in module 1, namely the symbolic-representational activity, with emphasis in the symbolic representation of known quantities; **Module 3** explored symbolic representation of known quantities in the context of relations of difference and equality, and finally **Module 4** proposed the same relations previously explored to be reconsidered with unknown quantities. Each of the four modules is described more clearly in the next session, with results issued from clinical analysis of videotaped sessions.

## 3. Modules and results

### 3.1. Module 1: *the magic machines*

This first module was composed by 14 situations about "magic machines" (Anno, 1991; Lieberman, Wey e Sanches, 1997), as briefly shown in Table 1 below. The unifying activity for the whole set of situations was identifying and expressing in natural language what kind of *transformation* was produced by each machine proposed, considering an input (what was introduced in the machine) and an output (what has been transformed after the passage by the machine; an example of machine, with a fragment of protocol is reproduced in Figure 1). The children were invited to work in small groups of four participants, these group being formed respecting previous relationship patterns in the classroom (a map of social interactions was previously built in order to guide the constitution of theses groups). The activity in small groups was always closed with general discussion, involving the whole class. Fourteen situations, corresponding to different machines, were proposed to the small groups; for each machine, three tasks were systematically proposed: (1) Trying to find out which transformation is performed by the machine, exploring also what would happen in case of inverse transformation: re-entering backwards the output (transformed object) in order to come back to the initial state of the transformation; (2) Composition of transformations, with re-entering of outputs when this action is considered as possible (e.g., two objects ➔ *four* objects, four objects re-entered ➔ *eight* objects). (3) Written production in natural language of what they have discovered about each machine (except situations 13 and 14). Main goals of module 1 were accomplished, as suggested by data in Table 1. The group of students was able to make sense of the activity negotiated with them (finding out the transformation performed by each machine, expressing this transformation in general terms), what was verified not only in terms of their achievement for each machine proposed, but also in terms of the achievement in tasks 13 and 14.

| Machines | Main aspect explored | Support needed by the students |
|---|---|---|
| **1. Putting noses and eyes in everything** | CPA | A |
| 2. Changing black-white things into colored ones | CPA | H |
| 3. Changing adult animals into young ones (a "youth machine") | CPA | H |
| 4. Giving names (in written language) to things | MAS | A |
| 5. Taking away the color of things (making them black & white) | CPA | A |
| 6. Transforming a set of a certain number of objets into a set of the same number of small and identical circles | MAS | H |
| 7. Transforming objects into the first letter of their usual names in Portuguese | MAS | H |
| 8. Transforming objects into letters (x, y and z) without relation with the names of the objects in Portuguese. | MAS | H |
| 9. Adding one more object to input set. | PMO | A |
| 10. Multiplying by two an input quantity of objects. | PMO | H |
| 11. Subtracting two objects from an input set of objects. | PMO | A |
| 12. Organizing the elements of a set in a certain order, according to an identified criterion. | *Deducing ordering criteria* | H |
| 13. Considering the description of cultural machines (*e.g., a balance*), the group must propose inputs with respective outputs. | *Many possibilities* | H |
| 14. The group is divided in smaller groups and invited to propose machines, with inputs and outputs | *Many propositions* | H |

**Table 1:** Machines proposed in module 1 with their characteristics and level of assistance demanded for the identification of the transformation done. [For the second column: **CPA** = changing a perceptual aspect of the input-object; **MAS** = making a symbolic attribution (names, letters); **PMO** = performing a mathematical operation. For the third column, **A** = achieved by the group without help, **H** = achieved with help from the teachers].

However, some important remarks must be taken into account: firstly, the proposition of *written* expression (phrases and/or words) of the general principle of each machine was hardly attained; because of this, the request of a written expression of machine principles became less explored, since the difficulty of written expression was not connected to difficulties in the comprehension of the very principle of each machine. Secondly, the interaction among students in each small group was only established with the intervention of the teachers, even though they had all accepted this way of working during the previous negotiation of the work contract; students in each group tried to interact basically with the teachers, frequently addressing colleagues in order to make fun and disturb them. The protocol transcription on the next page (Figure 1 plus dialogue) illustrates this pattern of work. We could observe an interesting work inside the small groups (no matter the difficulties mentioned above), and these gains could be reinforced during the discussions with the whole group. This module seems to have successfully furnished activities addressing important aspects of the conceptual field (Vergnaud, 1990) of algebra: the ideas of modeling, process / transformation and function.

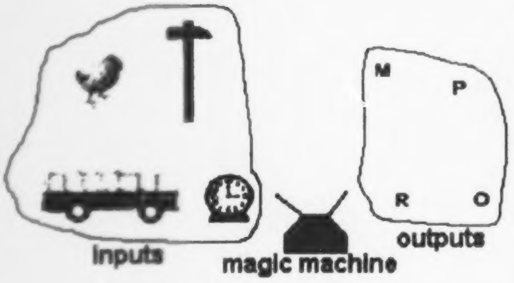


**Figure 1**: magic machine 7 (picture modified from Anno, 1991) [T= teacher; **S1** and **S2**, students from a group].

**T** – *So, what does this machine do?*
**S1** – *It sends things to here* [points to the right side of the picture] *forming letters* [She stands up and goes inspecting what is happening in another group].
**T** – [Calling S1 back] *What do you mean? Come back and explain! It forms what?*
**S1** – [Accepting to come back] *Letters! The chick* [Pinto, in Portuguese] *becomes P*; *the hammer* [Martelo] *passes through the machine and becomes M. The bus* [Ônibus] *becomes O.* [Now he is talking to another student, S2].
**S2** – *This one* [pointing to the hammer, Martelo] *becomes this letter* [pointing to O].
**S1** – *Not at all! It is* ***ma*** [from martelo]. *I is this one* [M].

### 3.2. Module 2: from *icons to symbols in the representation of quantities*

This module tried to explore further an activity firstly explored in module 1, namely the symbolic representation of cardinality. The work in this module started then with a presentation-sheet, in which a picture of a group of children is more and more reduced in its pictorial information, until the representation of the cardinality by five similar and very simple circles (see Figure 2 below).

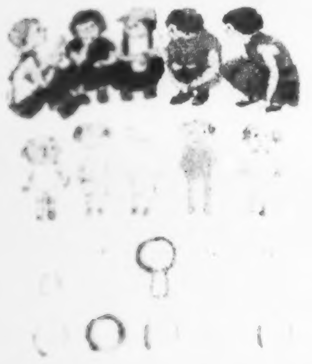


**Figure 2**: simplified representation of the presentation-sheet used in module 2 (illustrations modified from Anno, op. cit.).

Based on this initial idea, the students received three other work-sheets for individual work following this same idea of simplifying information until the representation of the cardinality of a given set of objects. The following activity (work sheet 4) tried to propose a representational problem to be solved in the context of the whole group: the representation of many sets of different elements with the same cardinals, as reproduced in figure 3 on the next page. The students were able to solve this problem by the proposition of different icons (squares, stars, etc.) to different sets of elements. Afterwards, the teacher proposed the use of *letters* and numeric coefficients as a way to represent the elements of each set (e.g., 5e for the set of five elephants).

Videotape analysis of activities developed by the students showed that the main goals of this module were fulfilled. Three important aspects must be emphasized in terms of new ideas developed by the group of students: 1) a known quantity can be represented with a conventional symbol; 2) different things must be represented by different symbols; 3) once a symbol is chosen for a certain thing, this choice must be kept for a certain mathematical task; on the other hand, it is important to keep a record of relations between chosen symbols and things being represented in order to avoid work-memory overload. The following protocol transcription illustrates these points:

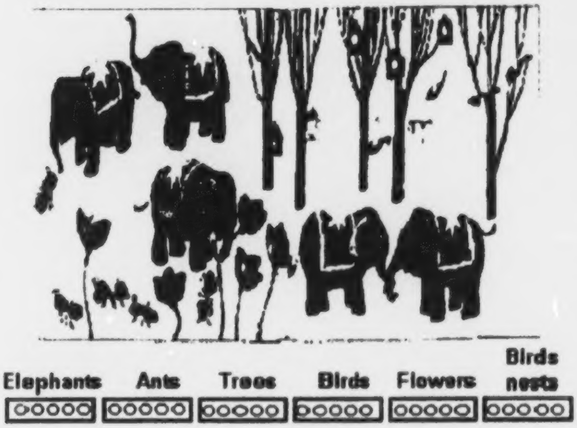


**Figure 3:** a problem in the simplified representation of cardinality using the same icons for sets of different things with the same quantity of elements.

**T:** [Showing the work sheet of the student S1 to the whole group]: *You see, for each thing S1 has chosen a different representation: small hearts to represent the elephants, balls for ants, small flowers to represent the trees. Now, think about this new work sheet: it is different of previous one* [The difference is the absence of legends with the names of sets of animals]. *What is the difference?*

**Ss:** *The names!*

**T:** *Pay attention, then. The names are lacking, aren't they? So, if the names are not written in the figure, we are going to have difficulties in remembering what set has been represented. So, what could we do in order to make things easier?*

**S2:** *To put names!* [coming back to previous situation].

**T:** *Yes, it is possible. One more idea?*

**S3:** *Make a drawing of each thing!* [in each set of small circles representing the respective quantity].

**T:** *Yes, we could make drawings, but it would take a long time. Look at S4's work: she put different symbols for different things! Go on, take your work sheets and try to propose something that help us to keep a track of what we have done, right?*

### 3.3. Module 3: *modeling relations of inequality and equality*

This module put emphasis in the modeling of relations between quantities, using the formal symbols of inequality (> and <) and equality (=). In this context, we tried to propose situations in which the students could pass from relations of inequality to equality (and vice-versa), through the consideration of the difference between two different quantities: if $A > B$, then $A - B = D$ (difference), therefore $A = B + D$ and $B = A - D$. This activity was developed in three steps, in the context of a general classroom activity: firstly, the students were asked to compare identical recipients with different or equal volumes of a colored liquid, using in this comparison the formal symbols >, < and = (a reproduction of part of individual work-sheets for this task can be seen in figure__ below ); secondly, the group was asked to find out and express orally the difference between the volumes of liquids being compared, finding out the volume of liquid to be taken away from one recipient, or added to the other, in order to obtain two equal volumes of liquid in both recipients (for this task, the recipients had volume-scales); thirdly, after having labeled the recipients with letters, the students were asked to express, in written language and in individual work-sheets, the relation between the volumes of two recipients (A > B, for instance), and also the way to make them equals in terms of measures of volumes (A = B + D or B = A - D , the letter D representing the difference of volume between the two recipients). The students had little difficulty in the step 1; the symbols of inequality and equality were put on the wall, in the classroom, and were used correctly in the comparisons of volumes of liquid. In step 2, however, in which they had to consider a certain difference using the measures of volume indicated in the scale of recipients, they had important difficulties. Because of this, the use of continuous quantities (liquids) was substituted by discrete quantities (small balls in the same recipients), as illustrated in figure 4 below: With this new representational support, we came back to step 1, and this time the task proposed in step 2 could be clearly understood and accomplished by the group. The protocol transcription below illustrates this point (T= teacher; G = group of four students; **Student 1**: a student from this group).

| **Teacher** | **Group of Students** |
|---|---|
| *- How many balls does Juliana have?* | *- Ten!* |
| *- And how many balls does Suzane have?* | *- Eight!* |
| *- Do the two girls have the same amount of balls?* | *- No!* |
| *- Who has more balls?* | *- Juliana!* |
| *- And how many extra balls does Juliana have?* | **Student 1**: [counting with the help of fingers] *Eight... she* [Suzane] *must take two balls.* |
| *- Ah! So, how many extra balls must Suzane take in the bag in order to have the same amount of balls of Juliana?* | **Student 1**: *Two.* |

In step three the students clearly need help from the teacher in order to express differences between discrete quantities. They were able to express differences orally, as illustrated by the two fragments above, but the passage to written expression only was obtained after clear suggestion of the teacher, as illustrated below (continuation of the same episode reproduced above):

T: *Ok, write in your work sheet: D = 2; let's put in the paper what you are saying! If we want that Suzane has the same amount of balls...* [the teacher writes in her sheet of paper] S = J . *Well, if we want this to be true, we must add two balls* [S = J + 2].

As suggested by data analyzed, the students were able to make use of what we have called the theorem-in-action (Vergnaud, 1990) of the passage from inequality to equality: *we can equalize two different quantities by adding the difference between them to the smaller, or subtracting this same difference from the bigger quantity.* Nevertheless, the symbolic expression of this theorem only could be attained in the context of interaction with the teacher.

### 3.4. Module 4: *modeling relations between quantities using numbers and letters*

The main goal of this set of activities was the treatment of situations concerning differences between *unknown* quantities (A > B), and the passage from difference to equality through the addition or subtraction of the difference, this difference being represented by the letter D (A = B + D or B = A - D). This module was the last one to be worked out in classroom, taking three weekly sessions from November to the end of the Brazilian school semester. We started with an activity-link with the previous module: the students were invited to compare two plastic boxes containing different an unknown quantities of small balls (the same deposits used in previous activities of module 3), these deposits being covered with paper in order not to allow visual inspection of the quantity of balls inside them. Two students were invited to hold the deposits in front of the group; each of these two students was "marked" by an icon of a happy face (☺) for the one owning more balls, and a sad face (☹) for the other owning fewer balls (the icons were drawn in sheets of paper and put on the floor, in front of each student). It was then proposed to the whole group that we would represent the unknown quantity of balls by a *letter* (they suggested to adopt the first letter of the name of the student holding the plastic box). All the group of students was able to model a relationship using letters to represent the quantity of balls in each box: G > J (session 1 of module 4) The passage to the equality considering the difference (G = J + D or J = G - D) was not easy even for the group that had been able to work previously with this same expression using numbers (module 3). In fact, two of the six groups were not able to understand this new situation, even though they were all able to understand the theorem-in-action involved in the passage from difference to equality, explored in module 3. This comprehension allowed some of them to pass from number relations to letters relations, but this passage was only possible with the help offered by the teachers, as illustrated by the dialog below [Module 4, sessions 2 and 3]:

> **T:** *Today we are going to work with our familiar plastic boxes again. Last week, we could compare boxes with different amount of balls, using numbers and letters, but we always knew the number of balls in the boxes. Today we are going to work with these boxes covered with paper, in order not to allow anyone to know the amount of balls in each box. Now, look at these boxes* [he holds a box in each hand facing the whole group]. *We don't know the quantities of balls, but we know that there is a difference between them. How could we write a sentence, like we did before, in order to express the difference between the boxes?* [pause]
> **S1:** *Using letters?*
> **T:** *Good idea! Come to the black board and show us what you are proposing.* (S1 writes: **T – C = D**)
> **T:** *Good! Now, explain what you have done.*
> **S1:** *T is bigger... T has more balls* [he points to the box with more balls]
> **T:** *And why you decided to use letters to write this sentence?*
> **S1:** [don't answer the question, and start drawing in the blackboard]
> **T:** Can anybody tell us why using letters in this situation?
> **S2:** *The first box had more balls then the second, and the first box is T, then ...* [he stops].
> **T:** *Did S1 know the quantity of balls in each box?*
> **Group:** [crying aloud]: *No!!!*
> **T:** *You see, in this case, when we don't know the exact amount of balls, we can use letters, and we can write the same sentence we wrote before with numbers! We don't know the quantities, but we know that there is a difference*

The group seemed to have understood this point, but only two of the six groups went further, modeling the most difficult problem of this module; this problem proposed a relationship between unknown quantities for which any empirical correlates (like balls, even not visible in plastic boxes) were offered. These data show that the passage from numerical relation to literal relation based on empirically detectable but verifyable quantities, and finally to unknown and not veryfiable quantities, represent different moments of psychological development of algebraic schemes. The goals of this module were partially reached, since many students (four groups in six) could not arrive to this point at all.

## 4. Concluding remarks

The set of data collected in the context of the present research allows us to propose that is possible to offer algebraic activity since very early in elementary level of formal education. The modeling of processes and transformations, as well as the passage from iconic to symbolic representation of known and unknown quantities, were aspects of the conceptual field of algebra undoubtedly explored in the context of the first two modules. In the other hand, difficulties in modules three and four confirm the difficulties in symbolic representation of unknown quantities, specially when the students are asked to express in written language relations of inequality where the explicit consideration of the difference is necessary for the passage to equality relations. These difficulties, however, must be reconsidered in other school contexts, since the particular and pervasive difficulties of northeastern Brazilian public school can have amplified them. An important extension of this research would be the verification of effects in school achievement, specially in terms of algebra instruction, a very important source of school failure in mathematics in Brazilian school

## References

Anno, M. (1991) *Jeux mathématiques* (vols. 1 e 2) Paris, Père Castor Flammarion..

Bodanskii, F. (1991). The formation of an algebraic method of problem-solving in primary school children. In: Davidov, V.V. (1991) *Soviet Studies in Mathematics Education: Vol. 6. Psychological Abilities of Primary School Children in Learning Mathematics* (pp. 275-338). Reston, NCTM.

Brito Lima, A.P., Da Rocha Falcão, J.T. (1997) *Early development of algebraic representation among 6-13 year-old children: the importance of didactic contract*. Proceedings of the XXIth International Conference for the Psychology of Mathematics Education PME-21). Lahti (Finlândia), julho-1997.

Brito Lima, A.P. (1996) *Desenvolvimento da representação de igualdades em crianças de 1ª. 6ª. série do 1º. grau*. Unpublished master thesis, Recife (Brazil), Universidade Federal de Pernambuco.

Cortes, A., Vergnaud, G., Kavafian, N. (1990) *From arithmetic to algebra: negociating a jump in the learning process*. Proceedings of the XIVth International Conference of the Psychology of Mathematics Education. Mexico.

Liberman, M., Wey, R.L.M., Sanchez, L.B. (1997) *Fazendo e compreendendo matemática*. São Paulo, Solução.

Lins Lessa, M.M. (1996) *Balança de dois pratos e problemas verbais como ambiente didáticos para iniciação à algebra: um estudo comparativo*. Unpublished master thesis, Recife (Brazil), Universidade Federal de Pernambuco.

Piaget, J. , Garcia, R. (1983) *Psychogenèse et histoire des sciences*. Paris, Flammarion.

Rashed, R. (1984) *Entre l'arithmétique et l'algèbre : recherches sur l'histoire des mathématiques arabes*. Paris, Les Belles Lettres.

Vergnaud, G. (1990) La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques*. 10-23, 133-170.

# TOWARDS A DEFINITION OF ATTITUDE: THE RELATIONSHIP BETWEEN THE AFFECTIVE AND THE COGNITIVE IN PRE-UNIVERSITY STUDENTS

Katrina Daskalogianni and Adrian Simpson
Mathematics Education Research Centre, Warwick University, Coventry, UK.

*This paper explores the relationship between beliefs about the nature of mathematics, previous experiences of mathematical situations and the initial behaviour of students on being confronted by a new mathematical problem. From the consistency of data provided by pre-university mathematics students we show that the affective aspects of an individual's mathematical thinking strongly influence their behaviour. In doing so, we suggest a new definition of the concept 'attitude'.*

## INTRODUCTION

While a number of researchers use the concept of attitude in their work, they often beg the question of what it is – almost reducing it to the pseudo-definition 'attitude is what attitude questionnaires measure', worthy of Wittgenstein at his worst! (a view explored in more detail by Lalljee, Brown and Ginsburg, 1984)

In the study upon which this paper is drawn, the startling consistency between beliefs about mathematics and behaviour in mathematical situations leads us to build further on Ajzen's "disposition to respond favourably or unfavourably to an object, person, institution or event." (1988, p. 4) and develop a new working definition of 'attitude' with important implications for teachers and researchers.

From the wide range of work in this area (such as Minato & Yanase, 1984; Ma & Kishor, 1997; Meyer and Eley, 1999) we can see that thinking mathematically depends intimately upon the affective aspects the situation for the given learner. Therefore, research into cognitive aspects of mathematics should take account of affective factors, such as beliefs, and emotions.

We see these factors as holding different roles according to the degree to which they influence cognitive responses. Research on beliefs in mathematics education focuses mainly on beliefs about the nature of mathematics (Ernest, 1991) and the studies that report the cause-and-effect relationship between affect and cognition examine the link between students' beliefs and mathematical performance particularly in problem solving situations (Schoenfeld, 1989; Silver, 1985).

Within the considerable quantity of research on the nature of affect in mathematics education (an excellent review is provided in McLeod, 1990), only a small amount addresses the connection between emotions and cognition (such as Skemp, 1979 and Mandler, 1989). Leventhal (1981) notes that emotion informs us about the capacity of different types of action. Thus, emotional reactions of students have an effect on their cognitive processes and consequently on their work in mathematics. McLeod et al. (1990) studied the emotional responses of students while solving problems and their findings suggest that their emotional responses are in a continuous state of change during the problem solving activity. An exploration of these research fields begs the question of the connection between them: how does affect influence behaviour?

The data presented in this paper focus on the relation between affect evidenced by espoused beliefs and experiential emotions, and cognition evidenced by mathematical behaviour.

## THE STUDY

The study reported here concentrated on the beliefs and mathematical behaviour of pre-university mathematics students. This is only the first part of a larger study investigating the connection between the affective and cognitive in doing mathematics through the transition from school to university.

The students who participated in the study had each been offered a place at a well-regarded UK university to study for a mathematics degree. Semi-structured interviews were conducted with them three months before their A-level examinations. Just prior to this, each participant completed a questionnaire. This included questions concerning their beliefs about mathematics and asked them to rank a number of statements about the nature of mathematics. At the end of the interview, a mathematical problem was given to the participants. The problem, adapted from Mason, Burton & Stacey (1982), was "A four digit palindrome is always exactly divisible by 11. Is that true?" This was designed to be accessible without the need for specialist mathematical knowledge, but was open to a number of different approaches that demonstrate different predispositions to mathematical behaviour, as we shall see.

## LINKING AFFECT AND BEHAVIOUR

The exploration of the affective is not an easy task from a methodological point of view (Kulm, 1980, Leder, 1985, Oppenheim, 1966). However, in analysing students' responses to the questionnaire items and their interview responses we obtained two perspectives on this aspect. Alongside their behaviour on the mathematical problem, this gave us three different forms of data we could explore. What is remarkable about this study is the level of consistency across these three forms showing a clear link between the beliefs and emotions espoused in the questionnaire and interview, and the behaviour identified in the work on the problem.

## EXAMPLES

In this section we examine, as examples, the questionnaire results, interview responses and initial problem solving behaviour of two students: Kathy and Sudir.

*Kathy*:

Kathy's response to the questionnaire ranked "mathematics is abstract" second and "mathematics is about putting things together", third[1]. Her view that mathematics is about synthesising tasks and putting things in an order dominates the interview:

> K: Generally I really like the algebra more… and I like fiddling around with equations.
>
> I: Why do you prefer algebra?
>
> K: I don't know. It's just…looking at it it seems a lot simpler to me, for me to just sort out everything and you can, with equations you can just…see where they're going. I find it easier to see what's happening with the, um, when it comes to like geometry and stuff and I'm thinking in other planes I find it a lot harder. …Um, sometimes you feel that you've kind of fudged a way through it and made things happen and you're just not sure whether you've actually followed on the logic fully or you've just kind of made it happen 'cause you wanted to… Especially in areas where I'm not so confident and I'm not so sure of my thinking, uhm, I become quite unsure of following my logic.
>
> I: What do think you gain from learning mathematics?
>
> K: Um, I think I've gained a more logical reasoning…
>
> I: Right, yeah…do you mean by logical reasoning?
>
> K: Um, just following on from what you know and, uhm, making assumptions, but making them based on what you know, so that they're relevant.

When Kathy starts working with the problem, she initially produces an algebraic representation of a four-digit palindrome *abba*. However, when she tries to carry on with the solution algebraically she struggles to work with this representation, writing:

*If ab is divisible by eleven then abba is*

$ab/11 = c + x/11 \quad 0 \leq x < 11$

$ab = 11c + x$

We suggest that Kathy starts to approach the problem algebraically because this is the way she believes a problem should be approached and this is the best way for her to "sort out everything".

Kathy begins by "fiddling around with equations" although she can't provide any explanation of her starting equation when asked. She then continues doing calculations from her initial algebraic representation, "so that they're relevant". However, her algebra-oriented approach, although not working initially, was used once again, successfully this time, when asked to think of a way of representing a palindrome in general. We believe that this is evidence of the extent to which her beliefs about mathematics have an effect on her behaviour in a given mathematical problem.

---

[1] Most of the students gave the statement "mathematics is about problem solving" the highest rank. The interviews showed, however, that there was a range of interpretations of this: from the routine completion of questions to the exploration of a mathematical situation. Therefore we distinguish the students by their next highest ranked statement.

*Sudir:*

Sudir on the other hand ranks the statement "mathematics is about procedures and computations", second in the questionnaire. A synopsis of his interview also shows that he believes mathematics should be numerical and closely related to real life.

> I: Which one is your favourite of all these [modules]?
>
> S: Um, Statistics!
>
> I: Could you tell me why?
>
> S: Um, I don't know. I enjoy…at home I'm doing quite a bit in terms of F1 scores or any little competitions. I just jump down all the different statistics and then I just, with Formula scores I just try to predict what's going to happen next. And with the World Cup Cricket I've entered a team, I just signed in Times newspaper!
>
> I: Oh really? Well, what about maths, the Pure Maths?
>
> S: Um, yeah, I find it all right […]. Sometimes I don't see the Pure Maths relevance to life, probably I can understand it and everything, but I don't see how to use it in everyday life [...] It does happen sometimes. I mean the majority of the time I don't understand the stuff, but I mean there will be some topics here and then, which I just plug in all the numbers into the calculation and just come up with a number and that's it!

Sudir's initial approach to the problem is to use his calculator in order to find a numerical pattern. He says, "at first, when I'm trying to prove something, I just look at the numbers first and see if there is a case generally." His beliefs about mathematics form the basis of a very practical and numerically oriented approach to a mathematical task. Even later when he tries to produce a form of a general proof, it is based on a numerical pattern of only two examples, which happens not to be correct. When Sudir was asked how he found this exercise he said:

> S: A bit weird! [laughs] It doesn't look like something we have done by now in terms of A-level maths. It's a bit abstract you could say.

Sudir's previous experiences with mathematical tasks have been successful by using a specialisation with the aid of a calculator and then a generalisation from his numerical results. In addition, he believes that mathematics is practical.

### GENERALISING

In both of these cases we see a strong relationship between the two aspects: the beliefs about mathematics evident from the questionnaire and the interview, and the behaviour exhibited in their initial work with a mathematical problem.

Table 1 summarises these aspects for the remaining students in this small study. It seems to indicate that this relationship is consistent across all of the students.

| Student | Highest ranked questionnaire statement and selected interview comments | Mathematical Behaviour |
|---|---|---|
| **Thomas** | Mathematics is about procedures and computations.<br>"I find maths is, is really about how clever you are."<br>"I can just blast off maths." | Attempt to apply an already known 'procedure': proof by induction, but he can't move on. He just wants to "get rid" of the problem. |
| **Derek** | Mathematics is about putting things together.<br>"I like proving, I like using what you already know again."<br>"Mathematics is solving problems…and using the tools as to find the solution to the problem."<br>"From doing maths you learn, um, I wouldn't say logically, but set things in a certain order." | Very analytical approach.<br>He separates cases: first a special one (aaaa) and then the general one (abba). |
| **Michael** | Mathematics is about procedures and computations.<br>"I saw maths as a subject more controllable than sciences."<br>"At maths there is always a definite answer you know you're working towards." | He provides an immediate algebraic represntation (abba = 1000a + 100b + 10b + a). |
| **Sandra** | Mathematics is about putting things together.<br>"I like numbers…use and calculate numbers!"<br>"[From learning maths] I gain the skills to do problem-solving, things like that, even if I thought that problem-solving was an awful thing to do!" | "Oh, dear! This is proof! Or counter-example kind of thing!"<br>She produces an algebraic answer, but doesn't know what a and b represent (in abba) for a specific number. |
| **Louis** | Mathematics is about memorising rules.<br>"Mathematics is more of an exact science and exact solutions, exact answers."<br>"Mathematics is something you can model and control."<br>"[Mathematics] is being able to master the techniques." | Very structured algebraic approach.<br>Although he says that it could be solved numerically, he proceeds with an algebraic long division. |
| **Simon** | Mathematics is about structure.<br>" [Mathematics] is about getting some data and then get a model to imitate the real world. You do some model and get a general answer, a quite precise answer. Get through the model and then hopefully get the answer, which you can relate to the real world." | Although his initial thought is proof by induction he finally proceeds algebraically (abba = 1000a + 100b + 10b + a). |
| **Daryl** | Mathematics is about inventing new ideas.<br>"I wanted to study maths because it's a very precise science and that excited me."<br>"I think maths is all about understanding numbers after which numbers generate structure, after which you try to understand this structure, the idea of proving." | Very structured proof.<br>He perceives what he has to prove for the case of the four-digit palindrome and then he moves to a generalisation of an n-digit one. |

| | | |
|---|---|---|
| **Jason** | Mathematics is about putting things together.<br>"Group Theory to me isn't the kind of maths I find interesting. It's more…I mean at the level we study it it certainly, it seems more vague. It's less precise let's say than matrices or numerical methods, something like that." | Very flexible and analytical thinking.<br>He first attempts proof by induction and he continues with proving nkkn = n00n + kk0 by testing each case separately. |
| **Dave** | Mathematics is about putting things together.<br>"I like proving identities and things like that, when you've got some equations like that equals cosx, 'cause you get some satisfaction by actually reaching the answer. It's nice to think, "Oh, I've got that, that's right!"" | He provides an algebraic answer without trying any other approach. |

Table 1

## DISCUSSION

In the case of Kathy we can see the degree to which her beliefs about mathematics and previous experiences of it influence her way of solving the given mathematical problem. Kathy's initial views about the nature of mathematics can be summarised as the ordering of equations and smaller mathematical tasks. This, along with her successful previous mathematical experiences while approaching a task algebraically, results firstly in her particular way of approaching the palindrome problem by starting from an equation (even though it was not correct) and "fiddling around" with it. In Sudir's case the same implication can be made. Sudir's beliefs about the nature of mathematics include the relevance of mathematics to real life situations and associating mathematics with number and calculation. His previous experiences with it may have reinforced his beliefs and predetermine his approach to the palindrome problem through specialisation with numerical patterns even though, in the case, a more formal proof is needed.

Across all eleven students we see a general tendency for the behaviour to match closely the statements and interview extracts from which we can infer their beliefs.

It is in the mechanism of this commonality that we can draw out our meaning for 'attitude'. Ajzen's 'disposition' does not provide us with an explanation for how such dispositions are formed nor how they influence their behaviour. In this work we do see such a disposition but we can also see its close link to behaviour and can suggest an obvious pathway for its formation. Thus we can suggest a new working definition of 'attitude' which expands upon Ajzen's:

> Attitude is the amalgam of the emotional experiences of a topic and the beliefs about the nature of the topic, which leads to a predisposition to respond with similar emotions and similar expectations in similar experiential settings.

We note, therefore, that attitudes towards mathematics are formed by students' experiences in mathematics and their beliefs about mathematics itself. The repetition of mathematical situations and the emotions they experienced while being in them, along with what they believe about the nature of mathematics and they way they should work in it, forms a predisposition to respond in a new mathematical situation. This predisposition becomes activated in a mathematical task and influences their cognitive processes and consequently their mathematical behaviour while working in it. As such, it is vital that a teacher is aware of, and works on, students' attitudes to mathematics as much as the content of mathematics. However, in this research (and in the other works we have come across) we have not explored how an existing attitude (in the sense defined above) comes to *change*. From a teacher's perspective this is an essential piece of work still to be done.

## REFERENCES

Ajzen, I. (1988). *Attitudes, Personality and Behaviour*. Milton Keynes: Open University Press.

Ernest, P. (1991). *The Philosophy of Mathematics Education*, London: Falmer Press.

Kulm, G. (1980). Research on mathematics attitude. In R. J. Shumway (*Ed.*), *Research in mathematics education*, 356-387. Reston, VA: National Council of Teachers of Mathematics.

Lalljee, M., Brown, L.B. & Ginsburg, G. (1984). Attitudes: disposition, behaviour or evaluation? *British Journal of Social Psychology*, Vol. 23, pp. 233-244.

Leder, G. C. (1985). Measurement of attitude to mathematics. *For the Learning of Mathematics*, Vol. 5, No.3, pp. 18-21.

Leventhal, H. (1981). The integration of emotion and cognition: A view from the perceptual-motor theory of emotion. In M. Sydnor Clark & S. T. Fiske (Eds.), *Affect and Cognition: The Seventeenth Annual Carnegie Symposium on Cognition*, pp. 121-156. Hillsdale, NJ: L. Erlbaum Associates.

Ma, X. & Kishor, N. (1997). Assessing the relationship between attitude toward mathematics and achievement in mathematics: A meta-analysis. *Journal for Research in Mathematics Education*, Vol. 28, No. 1, pp. 26-47.

Mandler, G. (1989). Affect and learning: Causes and consequences of emotional interactions. In D. B. McLeod & V. M. Adams (*Eds.*), *Affect and mathematical problem-solving: A new perspective*, pp. 3-19. New York: Springer-Verlag.

Mason, J., Burton, L. & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley.

McLeod, D. B. (1990). Research on affect in mathematics education: A Reconceptualisation. In D. A. Grouws (*Ed.*), *Handbook of Research on Mathematics Teaching and Learning*, pp. 575-596. New York: McMillan.

McLeod, D. B., Craviotto, C. & Ortega, M. (1990). Students' affective responses to non-routine mathematical problems: an empirical study. In Booker, G., Cobb, P. and Medicuti, T. (*Eds.*) *Proceedings of the Fourteenth Conference of the International Group for the Psychology of Mathematics Education*, Vol. I, pp. 159-166. Mexico: Mexico City.

Meyer, J.H.F. & Eley, M.G. (1999). The development of affective subscales to reflect variation in students' experiences of studying mathematics in higher education, *Higher Education*, Vol. 37, pp 197-216.

Minato, S. & Yanase, S. (1984). On the relationship between students attitudes towards school mathematics and their levels of intelligence. *Educational Studies in Mathematics*, Vol. 15, pp. 313-320.

Oppenheim, A.N. (1966). *Questionnaire Design and Attitude measurement*. London: Heinemann.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education*, Vol. 20, No. 4. pp. 338-355.

Silver, E. A. (1985). Research on teaching mathematical problem solving: Some underrepresented themes and needed directions. In E. A. Silver (*Ed.*), *Teaching and learning mathematical problem solving: Multiple research perspectives*, pp. 247-266. Hillsdale, NJ: Lawrence Erlbaum.

Skemp, R. R. (1979). *Intelligence, Learning, and Action*. New York: Wiley.

# A MEMORY-BASED MODEL FOR ASPECTS OF MATHEMATICS TEACHING

Gary Davis, University of Southampton, Southampton, UK

David Hill, Hordle Walhampton School, Lymington, UK

Nigel Smith, Twyford School, Winchester, UK

*We consider recent discoveries in brain function for declarative and procedural memories in the context of mathematical memory. These findings suggest to us a model for certain aspects of mathematics teaching, in which developing brains of children benefit from an external agent to assist in the formation of declarative memories from procedural memories. We consider some implications of this model for classroom teaching.*

## Introduction

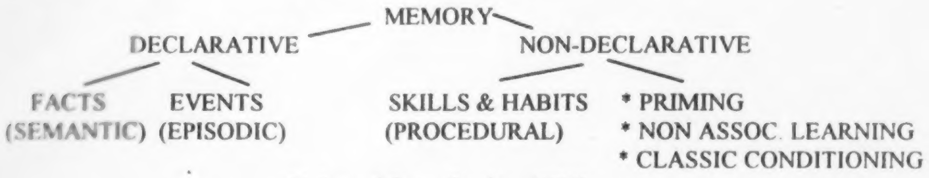
The nature of learning has been a constant area for research since the beginnings of modern psychology. Models for the activity of teaching, especially in the area of mathematics, have been much less common (see, however, Simon, 1995; Steffe and d'Ambrosio, 1995; and references). In this article we consider recent research into how human brains process declarative and procedural knowledge. On the basis of a simple but plausible assumption on the nature of developing brains we suggest that, at least in the early years of schooling, an external agent can greatly assist in the formation of declarative memories from procedural memories in mathematics. Given the importance of both procedural and declarative memories for mathematics, this suggests a significant function for a mathematics teacher: to fulfill that of an external conduit from one part of a student's brain to another.

## Declarative and procedural memory

*Declarative* memory (often referred to as *explicit* memory) is memory for facts and events, including scenes, faces and stories. Declarative memory depends upon having relevant brain structures intact: people with certain forms of amnesia, due to identifiable brain damage, lose their ability to function efficiently - or at all - with declarative memory (Squire, 1994, p.203). However, the concept of declarative memory is not due simply to disassociation of memory caused by brain damage: recent findings (for example, Eichenbaum, 1994; Ullman *et al*, 1997) demonstrate that specific areas of the brain are dedicated to declarative memory (see also, Bransford *et al*, 1999, p. 112 *ff*.)

Declarative memory includes episodic and semantic memory - a distinction due to Tulving (1983). This distinction is not sharp since in practice many memories involve both episodic and semantic aspects. The essential difference is that semantic memories are relatively bereft of context: they appear to the rememberer simply as known facts, whereas episodic memories involve significant components of context in which the memory was formed. The distinction in mathematical understanding is therefore critical because it relates to memories that have been pared down so as to be apparently free of context or irrelevant detail. In mathematics, episodic memories can facilitate the establishment of semantic memories (Davis, 1996).

Procedural memory is one form of non-declarative memory although the two are often identified as being the same. Cohen and Squire (1980) coined the term *procedural memory* for the ability to learn sensorimotor tasks in the presence of other severe memory losses. They postulated a memory for *how* to do things as distinct from a memory for *what* was done or what was recalled as a fact.



(Adapted from Squire, 1994)

### Declarative and procedural memory in mathematics

The distinction that is commonly made between declarative and non-declarative memory is particularly pertinent to learning mathematics. We take it to be the case that declarative memory - at least certain forms of it - in mathematics is a higher and more useful form of memory than procedural. This is because, in mathematics, we see declarative memory (memory *that*) as a form of extraction or compression of the essence of a procedure or procedures. Declarative memories in mathematics seem to associated with the formation of mathematical schema and the processes of encapsulation of procedures (ref. Tall *et al*, to appear). Indeed the original ancient Greek meaning σξεμα was the "essence" of a thing: its characteristic nature.

A prime example of declarative memory can be observed in Maher and Speiser's (1996) account of a young student who related binomial expansions to memories of building block towers. In using her memories of building block towers to establish a potent link between these towers and binomial expressions this student was able to *declare that* the binomial expressions were "just like" the towers. She effectively stopped carrying out the procedure of expanding the binomial expression $(a + b)^3$ and declared that it was just like building towers of height 3 from two colors of blocks.

Classic examples of procedural memories arise from students' engagement with taught algorithms such as long division, multiplication of whole numbers, the Babylonian iterative method for finding square roots, the bisection algorithm for finding square roots, and Euclid's algorithm for the greatest common divisor of two positive integers. It is a commonplace observation that many students learn to carry out these procedures but have great difficulty talking about them in the absence of doing them.

An almost universal example of a non-taught procedure that is recalled procedurally very well by children as young as 3 and 4 years of age is sharing, also known as dealing or distributive counting (Hunting and Davis, 1991; Miller, 1984). However these same children are largely unable to talk about and contemplate the act of sharing as a fact (Hunting and Davis, 1991).

### Neurophysiology of declarative and procedural memory

A clue to the difficulty in articulating procedures in mathematics comes from work of Ullman *et al* (1997) on language difficulties in sufferers of Alzheimer's disease on the one hand and Huntington's and Parkinson's on the other. Their work suggests that word memory relies on areas of the brain that handle declarative memory - memory of facts and events. These appear to be the temporal or parietal neo-cortex. However, rules of grammar seem to be processed by areas of the brain that manage procedural memory, the basal ganglia, which are also involved in motor actions. Declarative memory has been considered in detail by Eichenbaum (see, for example, Eichenbaum, 1994) who locates it primarily in the hippocampal system. This area of the brain is part of the limbic system, responsible largely for emotion and drives. The hippocampal system has many connections both to and from the temporal and parietal lobes of the neo-cortex. The current view suggests that for declarative memory, episodic memories reside largely in the hippocampal region whilst semantic memories reside principally in the neo-cortex.

## A memory-based model for teaching mathematics

That there seem to be two distinct brain areas for procedural and declarative memory must make us suspicious. In mathematical settings, at least, the region devoted to declarative memory may have difficulty - that is, few mechanisms for - taking as its basic material the activities of the region responsible for procedural memory. If so, the role of teacher becomes even more evident: as an external conduit to allow declarative memories to be formed from the raw material of stored procedural memories.

A decisive force in the creation of mathematical schema may be an appropriate agent capable of externalizing procedural memory and utilizing it so that a student can form declarative memories, both episodic and semantic. The reason for this, we hypothesize, is that the temporal and parietal neo-cortex has, in young children, few mechanisms for taking the memory activities of the basal ganglia as raw data for the formation of new declarative memories. What has to happen, we suspect, is that an agent *externalizes* those memories of procedures from the basal ganglia and recasts them in a form suitable for the hippocampal region and/or the temporal and parietal neo-cortex to process them as declarative memories. Our model, building on what is known about brain structures for memory, has the following features:

1. There are, in young, developing brains, relatively few direct connections between the brain regions implicated in procedural memory and those implicated in declarative memory. There are known to be direct connections into the basal ganglia - responsible for motor control and planning - from all over the brain. We postulate that in younger brains there are few direct connections from the basal ganglia to the hippocampal region and/or the temporal and parietal lobes.
2. Episodic memory, as a form of declarative memory, resides largely in the hippocampal region.
3. Semantic memory, as a form of declarative memory, resides largely in the neocortex, particularly the temporal and parietal lobes.
4. An external agent can assist in the formation of episodic memory from procedural memory.
5. An external agent can stimulate pre-existing connections between the hippocampal region and the neo-cortex to assist in the transfer of episodic to semantic memory.

### Focus of attention and the central role of episodic memory

In the enhancement of mathematical memory the major issue for a teacher to focus on is assisting students to establish appropriate episodic memories of mathematical activity. In our model of teacher as external agent, there are three possible ways in which the teacher can play a part in enhancing memory. These are, assisting in the transfer of:

1. procedural memories to episodic memories;
2. episodic memories to semantic memories;
3. procedural memories directly to semantic memories.

Not a lot seems to be known about the *necessity* of semantic memory building on episodic memory (Squire *et al*, 1993). We hypothesize that a direct transfer of procedural to semantic memory is unlikely to occur in the domain of mathematics. Such a transfer would involve establishing a mathematical fact or facts from memories that involve only habit or the execution of procedures, in the absence of episodic memories. An example might be a student who is involved in the practical successive approximation square roots of whole numbers declaring that the square root of a number is another number whose square is the first, without the intermediation of episodic memory pertinent to the approximation of square roots. This simply does not gel with our experience: students will speak about their experiences first by using episodic memory. It seems to us, therefore, that episodic memory plays a major role in the transition of procedural memories to semantic memories. In the formation of such episodic memories a student's focus of attention is critical. This is because not everything about an episode is appropriate to recall for the purposes of obtaining a deeper understanding of the mathematics involved. The memory of it being a hot spring Friday afternoon in a class of 25 other students, all wearing jeans, is not relevant to the formation of episodic memories of factoring quadratic equations. What is relevant is a student's memory of what they were doing, in a mathematical sense, as they factored a particular quadratic.

### Implications of the model

Our brain-based model of teacher as external agent between an individual child's procedural and declarative memories has implications for the sorts of activities that a teacher engages in to facilitate the formation of declarative memories. The most obvious implication is for what is likely *not* to work. Although we regard practice at procedures and algorithms as important, our model indicates that this alone will not be sufficient for all, or even a majority, of students. The ability to form declarative memories is not going to arise simply from carrying out procedures.

What might a teacher do to assist a student to form declarative memories from memories of procedures? One clue comes from the student studied by Maher and Speiser (1996). As a child in elementary school, building towers with her classmates, she persistently and habitually, asked why: "why is it so?" For example, upon finding

16 towers of height 4, she and the other children eventually argued that each tower of height 4 came from a tower of height 3, and there were exactly 2 ways to build a tower of height 4 from any given tower of height 3. This was their *reason* that the number of towers doubled as the height increased by 1. This, for them, was now a declarative fact. These children essentially acted as external agents for each other in turning procedural memories into declarative memories. The drive came from demanding an answer to "why?". Asking why has to be an attractive proposition to a student: otherwise their answer to "why" may well be: "I don't know. And I don't care."

Another clue comes from the central role of episodic memory- as a halfway house in the transition from procedural to semantic memory. One way to assist in the formation of episodic memories, focussing on aspects of episodes pertinent to understanding mathematics, is for a student to elaborate their solution procedures to problems through explanation and discussion. A student cannot talk about their solution procedures unless they have something episodic or semantic to talk about. This is because procedural memory, by its very nature as a form of non-declarative memory, is not accessible to verbal discussion (Squire, 1994). Insisting that a student talk about their solution procedures therefore forces them to bring to mind episodes in which they carried out those procedures. In other words, it forces them to establish declarative memory for procedures in the form of relevant episodes.

Finally, another implication of our model is that teachers need to ask themselves what is the nature of their own memories and types of understanding. This is because, as Ma (1999) points out:

"... none of those teachers whose knowledge was procedural described a conceptually directed teaching strategy. ... Not a single teacher was observed who would promote learning beyond his or her own mathematical knowledge" (p. 54)

## References

Bransford, J. D., Brown, A. L. & Cocking, R. R. (Eds.) (1999) How People Learn. Brain, Mind, Experience and School. Washington: National Academy Press.

Cohen, N.J. & Squire, L.R. (1980) Preserved learning and retention of pattern-analyzing skill in amnesia: Dissociation of knowing how and knowing that. *Science* 210, 207-210.

Davis, G. (1996) What is the difference between remembering someone posting a letter and remembering the square root of 2?. In L. Puig & A. Guttieréz, (Eds.) Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education, Vol 2, pp. 271-280. Valencia: Universitat de València.

Eichenbaum, H. (1994) The hippocampal system and declarative memory in humans and animals: experimental analysis and historical origins. In D. Shacter & E. Tulving (Eds.) Memory Systems 1994, Cambridge, Mass: MIT Press.

Hunting, R.P. & Davis, G. (1991) *Early Fraction Learning*. New York: Springer Verlag.

Ma, L.-P. (1999) Knowing and Teaching and Elementary Mathematics. New Jersey: Lawrence Erlbaum.

Maher, C. & R. Speiser, R. (1997) How Far Can You Go with Block Towers? In E. Pehkonen (Ed.), Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education, Vol 4, pp. 174-181. Helsinki: University of Helsinki.

Miller, K. (1984) Child as the measurer of all things: Measurement procedures and the development of quantitative concepts. In C. Sophian (Ed.) Origins of Cognitive Skills, pp. 193-228. Hillsdale, N.J.: Erlabaum

Simon, M.A. (1995) Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2),114-145

Steffe, L.P. & d'Ambrosio, B.S. (1995) Toward a working model of constructivist teaching - a reaction to Simon. *Journal for Research in Mathematics Education*, , 26 (2), 146-159.

Squire, L. R. (1994) Declarative and non-declarative memory: Multiple brain systems supporting learning and memory. In D. L. Schacter & E. Tulving (Eds.) *Memory Systems, 1994*, pp. 203 - 231. Cambridge, Mass.: MIT Press.

Squire, L. R., Knowlton, B. & Musen, G. (1993) The structure and organization of memory. Annual Review of Psychology, 44, 453- 495. Reprinted in D. Shanks, Human Memory. A Reader. 1997, pp. 152 - 197. London: Arnold.

Tall, D.O., Thomas, M., Davis, G., Gray, E. & Simpson, A.P. (to appear) What is the object of encapsulation? *Journal of Mathematical Behavior*

Tulving, E. (1983) *Elements of Episodic Memory*. Oxford: Oxford University Press.

Ullman, M.T., Corkin, S., Coppola, M., Hickok, G., Growdon, J. H., Koroschetz, W. J. & Pinker, S. (1997) A neural dissociation within language: Evidence that the mental dictionary is part of declarative memory, and that grammatical rules are processed by the procedural system. *Journal of Cognitive Neuroscience*, 9(2), 266-276.

# INVOLVING PUPILS IN AN AUTHENTIC CONTEXT: DOES IT HELP THEM TO OVERCOME THE "ILLUSION OF LINEARITY"?

Dirk De Bock* **, Lieven Verschaffel*, Dirk Janssens* and Karen Claes*
University of Leuven* and EHSAL, Brussels**; Belgium

***Abstract.*** *Many authors have argued that gaining insight into the relationships between lengths, areas and volumes of similar geometrical figures is usually a slow and difficult process. Thereby, pupils are often misled by the so-called "illusion of linearity": they apply the ratio between the linear measurements also to determine the area and volume of an enlarged or reduced figure (NCTM, 1989). Pupils' misuse of the linear model in this context was recently demonstrated in a series of empirical studies (De Bock, Janssens & Verschaffel, 1998a, 1998b; De Bock, Janssens, Verschaffel & Rommelaere, 1999). The present investigation builds on these studies and explores the influence of authentic and realistic contexts and of self-made drawings on the illusion of linearity in this content area. Contrary to our expectations, the results did not reveal any beneficial effect of the authenticity factor on pupils' performance nor of the drawing activity.*

## Theoretical and empirical background

The different behaviour of one-, two- and three-dimensional quantities under enlarging and reduction operations is fundamental to measurement and critical to scientific applications. The principle governing this behaviour is well-known: an enlargement or reduction by factor $r$, multiplies lengths by factor $r$, areas by factor $r^2$ and volumes by factor $r^3$. A crucial aspect of understanding this principle is the insight that these factors depend only on the magnitudes involved (length, area, and/or volume), and not on the particularities of the figures (whether these figures are squares, cones, irregular figures, etc.). This principle is mathematically so fundamental that it must come first, both from a phenomenological and didactical point of view. As Freudenthal (1983, p. 401) states it: "This principle deserves, as far as the moment of constitution and the stress are concerned, priority above algorithmic computations and applications of formulae because it deepens the insight and the rich context in the naive, scientific, and social reality where it operates."

Many authors have argued that gaining insight into the above-mentioned relationships between lengths, areas and volumes of similar figures is usually a slow and difficult process. In the *NCTM Standards*, for instance, we read in this context that "... most students in grades 5-8 incorrectly believe that if the sides of a figure are doubled to produce a similar figure, the area and volume also will be doubled." (NCTM, 1989, pp. 114-115). In other words, pupils tend to use the linear scale factor instead of its square or cube to determine the area or volume of an enlarged or reduced figure. This misuse of linearity is a manifestation of the so-called "illusion of linearity", a misconception emerging in different content areas of science and mathematics education.

Recently, the strength of the illusion of linearity as well as its resistance to change were empirically investigated in four related studies by De Bock et al. (1998a, 1998b, 1999). Large groups of 12-13- and 15-16-year old pupils were administered the same set of proportional and non-proportional items about length and area of similar plane figures under different experimental conditions. The experimental items were formulated as traditional word problems and were built around different types of plane figures (regular figures such as squares and circles, and irregular figures). The next problem is an example of a non-proportional item about the enlargement of a square: "Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?".

The major results of these studies can be summarised as follows. First, the tendency to apply proportional reasoning in the solution of the non-proportional problems proved to be extremely strong in the age-group of 12-13-year olds, and was still very influential among 15-16-year olds: overall percentages of correct responses on the non-proportional items varied between 2% and 7% in the group of 12-13-year olds and between 17% and 22% in the group of 15-16-year olds (De Bock et al., 1999). Second, the type of figure involved played a significant role: pupils performed better on the non-proportional items when the figure involved was regular (a square or a circle), but as a drawback they performed worse on the proportional items about these regular figures because some pupils started to apply non-proportional reasoning on the proportional items too. Third, the provision of visual support (in the form of given drawings made on squared paper) and the provision of metacognitive support (in the form of an introductory task that confronted pupils with a correct and incorrect solution strategy to a representative non-proportional item and asked them which one was the correct just before the start of the actual test), yielded a significant, but unexpectedly small effect on pupils' performance on the non-proportional items; and, once again, this positive effect was accompanied by worse performances on the proportional items in the supported conditions. Fourth, the misuse of the proportional model proved to be partially caused by the formulation of the experimental items in a traditional "missing-value" format (a format in which the unknown has to be determined on the basis of three given numbers). In fact, pupils confronted with mathematically equivalent problems but rephrased as "comparison problems" (e.g. "Farmer Carl manured a square piece of land. Tomorrow, he has to manure a square piece of land with a side being three times as big. How much more time would he approximately need to manure this piece of land?"), resisted more easily the trap of proportional reasoning; but, once more, as a result, these pupils sometimes began to question the correctness of the proportional model for proportional problems. However, while the effect of problem formulation was significant, it was again still relatively small, as still more than half of the non-proportional items were solved erroneously in the "comparison condition".

While these previous studies revealed pupils' almost irresistible tendency to apply proportional reasoning in problem situations for which it was totally inappropriate, they still did not yield a satisfying answer to the question *why* so many pupils fell into this "proportional trap", even after receiving considerable support. The present study focuses on a new hypothetical explanation for pupils' unbridled proportional reasoning, namely the (lack of) authenticity of the problem context. Indeed, all experimental items used in the previous studies were traditional school word problems, built around farm life and other rather lean contexts having no special meaning or attraction to 12-16-year olds nowadays. Therefore, one could anticipate that pupils' performance on the non-proportional items would increase drastically if one would succeed in increasing the authenticity of these problem situations for the pupils. Some evidence for this assertion can be found in the work of scholars of the Freudenthal Institute, who have explored pupils' difficulties with the influence of linear enlargement on area and volume in realistic contexts (like "With the giant's regards" in Streefland, 1984, or "Gulliver" in Treffers, 1987). In his report on an experiment with sixth graders, Treffers states: "This question *[How many Lilliputian handkerchiefs make one for Gulliver?]* introduces the influence of linear enlargement on area. The pupils have no difficulty with the problem." (Treffers, 1987, p. 5). However, this statement is not empirically documented or supported in Treffers' report. To test empirically the facilitating power of making the problem context more realistic, the present study was executed.

## Method

Hundred-and-fifty-two 13-14-year olds and hundred-and-sixty-one 15-16-year olds participated in the study. In both age-groups, four equal subgroups were formed in which (1) different study streams were equally represented, (2) the average number of hours a week pupils spent at mathematics was the same, and (3) pupils' average result on the final 1998 mathematics examination was not significantly different. All pupils were confronted with a paper-and-pencil test consisting of 6 experimental items (2 proportional items and 4 non-proportional items) about the relationships among the lengths, areas and volumes of similar regular and irregular figures and some buffer items. This test was administrated differently in the four experimental groups. In the A–D– group (= no authentic setting and no drawing group) the experimental items were presented in the form of a series of non-related traditional school problems in a neutral context (e.g. "The side of square Q is 12 times as long as the side of a square R. If the area of square Q is 1 440 cm$^2$, what's the area of square R?"). In the A–D+ group (= no authentic setting but drawing group), the same test was presented to the pupils, but before any measure of the geometrical figure was given, they first had to draw a similar figure next to its original using a given linear scale factor. In the A+D– group (= authentic setting without drawing group), we presented a parallel version of the test, but pupils' involvement in the problems was experimentally enhanced by prefacing the test by an assembly of well-chosen video fragments telling the story of Gulliver's visit to the isle of the Lilliputians, a world in which all lengths are 12 times as small as in our world, the world of Gulliver, and all items were linked to these video fragments. For instance, rephrasing the square problem mentioned above, we asked "Gulliver's handkerchief has an area of 1 296 $cm^2$. What's the area of a Lilliputian handkerchief?". Finally, the pupils of the A+D+ group (= authentic setting and drawing group) received the same version of the test as in the A+D– group, but first had to make a drawing of the geometrical object in the Lilliputian world next to a given drawing of that object in Gulliver's world for every test item.

**Hypotheses**

First, in line with the results of De Bock et al. (1998a, 1998b, 1999), we predicted that the pupils' performance on the proportional items would be very high, while their scores on the non-proportional items would be very low (= hypothesis 1), and that the 15-16-year olds would perform better on the test in general and on the proportional items in particular than the 13-14-year olds (= hypothesis 2).

Third, as shown in many research in the field of realistic mathematics education (see, e.g., Streefland, 1984; Treffers, 1987), we hypothesised that plunging pupils into a rich and attractive context - in our case: the story of Gulliver's visit to the Lilliputians, coming alive in a fantastic video respecting faithfully the original 1:12 ratio for lengths - would help them to engage themselves in meaningful and thoughtful thinking processes about the mathematical problems based on this context, especially with the non-proportional items. Accordingly, we predicted for both age-groups higher scores for the two authentic-setting groups (= A+D– and A+D+) than for the two no-authentic-setting groups (= A–D– and A–D+), due to a positive effect of the authenticity factor on pupils' capacity to model and solve the non-proportional items.

Fourth, we anticipated a better performance on the non-proportional items for the pupils who had to make a drawing of a figure similar to a given original as an integral part of their solution process. Several studies report positive effects of drawings and diagrams on pupils' mathematical problem solving (see, e.g., Moyer, Sowder, Threadgill-Sowder & Moyer, 1984), that are even strengthened when these visualisations are made by the learners themselves (Aprea & Ebner, 1999). Accordingly, we predicted for both age-groups higher scores for the two drawing groups (= A–D+ and A+D+) than for the two no-drawing groups (= A–D– and A+D–), due to the anticipated positive effect of making a drawing on pupils' capacity to represent and solve the non-proportional items.

Fifth, we hypothesised a cumulative effect of the authenticity and drawing factor. Arguably, both factors are relatively independent and can act upon each other. For instance, a pupil watching the film may realise that it is impossible that areas and volumes in the Lilliputian world only measure one twelfth of those in Gulliver's world, but making a drawing may effectively help him to discover the exact value of the scale factor for areas and volumes. Therefore, we predicted that the best performance on the test, and on the non-proportional items in particular, would come from the A+D+ group.

## Analysis

All responses on the proportional and the non-proportional items were categorised as "correct" or "incorrect". No distinction was made between different types of incorrect responses in the present analysis.

The hypotheses were tested by means of a "2 × 2 × 2 × 2" analysis of variance with "Proportionality" (proportional vs. non-proportional items), "Age" (13-14- vs. 15-16-year olds), "Authenticity" (authentic vs. traditional setting) and "Drawing" (drawing vs. no-drawing) as independent variables, and the number of "Correct answers" as the dependent variable. A post-hoc analysis revealed that there turned out to be a problem with the comparability of the non-proportional items about irregular figures in the different experimental conditions and therefore, these items were left out in the present analysis. We also did additional quantitative as well as qualitative analyses of the nature of pupils' erroneous answers, their formal and informal solution strategies, and their schematic drawings, but, because of space restrictions, the results of these analyses will not be reported here.

## Results

Table 1 gives an overview of the percentage of correct responses for the four groups of 13-14- and 15-16-year olds on the proportional and the non-proportional items in the test.

| | 13-14- year olds | | 15-16-year olds | |
|---|---|---|---|---|
| | Proportional items | Non-proportional items | Proportional items | Non-proportional items |
| A–D– | 96 | 46 | 93 | 61 |
| A–D+ | 91 | 18 | 97 | 47 |
| A+D– | 99 | 25 | 97 | 40 |
| A+D+ | 96 | 0 | 97 | 33 |

*Table 1. Overview of the results*

The results provided a very strong confirmation of the first hypothesis. Indeed, the analyses revealed a strong main effect of the task variable "Proportionality" ($p < .01$): for the two age-groups and for four experimental groups together, the percentages of correct responses for all proportional and for all non-proportional items were 96% and 34%, respectively.

The second hypothesis was confirmed too: the factor "Age" had a significant main effect ($p < .01$): the 15-16-year olds performed better on all experimental items than the 13-14-year olds; percentages of correct answers were, respectively, 71% and 59%. Furthermore, the predicted "Age × Proportionality" interaction was found too ($p < .01$): while the 15-16-year olds answered nearly twice as much non-proportional items correctly than the 13-14-year olds (45% and 23% correct responses, respectively), the performance of the two age-groups on the proportional items was more or less the same (96% and 95% of correct responses, respectively).

Third, contrary to our expectation, there was no positive effect of the authenticity variable on pupils' performance on the test as a whole, and on their scores on the non-proportional items in particular. To the contrary, pupils who watched the video fragments and who received the video-related items performed even significantly worse than the pupils from the two other groups ($p < .01$) on the test in general (61% correct responses for the authentic-setting groups versus 68% for the other groups), and on the non-proportional items in particular (25% correct responses for the authentic setting groups versus 43% for the other groups).

Fourth, the results did also not support the hypothesis on the anticipated drawing effect. To the contrary, pupils who were asked to make a drawing of a geometrical figure similar to its given original, performed even significantly worse than pupils from the two other groups ($p < .01$) on the test in general (59% correct responses for the two drawing groups versus 70% for the two no-drawing groups), and on the non-proportional items in particular (23% correct responses for the drawing groups versus 44% for the non-drawing groups).

Fifth, and not surprisingly taking into account the outcomes of the statistical tests with respect to hypotheses 3 and 4, we did not find support for the hypothesis concerning a cumulative positive effect of the authenticity and drawing factor. Actually, as shown in Table 1, the results of the A+D+ group were by far the lowest of the four experimental groups.

Because of the totally unexpected nature of the results with respect to the impact of the experimental variables, we set up a replication study. However, because of practical reasons, this replication study involved only the manipulation of the drawing factor (and not of the authenticity factor), which we considered the most remarkable outcome of the study. Another difference with the previous study was that we guaranteed that all pupils in the drawing condition had enough time to complete the test. Indeed, it could be argued that the unexpected outcomes were possibly due to the fact that pupils who had to watch the video first and/or who had to make drawings before actually solving the word problems, had less time to solve these problems than the pupils in the no-authenticity and/or the no-drawing conditions, who could spend all the available time solving the problems. However, the results of that replication study were exactly the same, in the sense that the pupils who were instructed to make drawings performed considerable worse on the test as a whole and on the non-proportional items in particular, than the pupils from the no-drawing condition.

## Discussion

Studies by De Bock et al. (1998a, 1998b, 1999) convincingly demonstrated that the vast majority of pupils fail on problems about the effect of a similarity on area and volume, even when considerable help is provided (giving metacognitive support, presenting the problem in another format, ...). In the present study, we tested the effect of increasing the authenticity of the problem context and of making a drawing of the problem situation on pupils' performance in this content area. Contrary to our expectations, both experimental manipulations didn't yield the expected results; they even yielded a negative effect on pupils' performance. A replication study indicated that differences in available time for actually solving the problems cannot account for the remarkable results.

With respect to the diametrically opposite results for the authenticity effect, several hypothetical explanations can be formulated. First, the way in which the problems were stated and presented in the no-authenticity condition was more similar to the way in which these problems are formulated and presented during the normal mathematics lessons at school. Maybe the unfamiliar nature of the authentic setting elicited in the pupils from these experimental groups feelings of stress or fear of failure, which may have led to extra errors. A second possible explanation can be found in Salomon's (1981) account of the mediating effects of people's perceptions of media characteristics on their willingness to invest mental effort during learning and problem solving, and, consequently, on their performance. According to Salomon, students perceive video as a less difficult medium than written materials, and therefore are inclined to invest less mental effort in working with messages sent by this easy medium as compared to media that are perceived as difficult. For a third possible explanation, we refer to a related finding reported by Boaler (1994) that pupils - and especially girls - were likely to underachieve in realistic mathematical contexts as compared to problems presented in an abstract way. Boaler analysed the performance of 50 students from each of two schools - one school with a traditional approach and one with a reform-based approach to mathematics education - on two sets of three questions assessing the same mathematical content through different contexts. Interestingly, the girls from the traditional school attained a lower grade on an item built around the context of fashion, than on an isomorphic question embedded in an abstract context or in a context that was less appealing for these girls. According to Boaler, these girls' underachievement on the fashion item was caused or influenced by their greater involvement in the question, which led them away from its underlying mathematical structure. Although there are quite some differences between the way in which the authenticity factor was operationalised in both studies, it is possible that pupils' emotional involvement in the fantastic world of the Lilliputians may have had a negative instead of a positive influence on their mathematical performance.

We also have no convincing explanation for the absence of a positive effect of asking pupils to make a drawing, and even for the existence of a negative drawing effect. As such, this result is in line with what has been found in other studies, namely that making a sketch or drawing - like other heuristic methods - does not guarantee the problem solver to find of a correct solution, especially when the necessary domain-specific knowledge is missing (Van Essen & Hamaker, 1990). But, as far as we know, in none of these studies making a drawing had a strong negative effect!

In our future research we will continue our attempts to understand the bizarre results obtained in the present study, and to find ways to overcome the deep-rooted linear obstacle.

## References

Aprea, C., & Ebner, HJ.G. (1999, August). *The impact of active graphical representation on the acquisition and application of knowledge in the context of business education.* Paper presented at the 8th European Conference for Research on Learning and Instruction, Göteborg, Sweden.

Boaler, J. (1994). When do girls prefer football to fashion? An analysis of female underachievement in relation to 'realistic' mathematic contexts. *British Educational Research Journal, 20*, 551-564.

De Bock, D., Verschaffel, L., & Janssens, D. (1998a). The predominance of the linear model in secondary school pupils' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics, 35*, 65-83.

De Bock, D., Verschaffel, L., & Janssens, D. (1998b). The influence of metacognitive and visual scaffolds on the predominance of the linear model. *Proceedings of the 22th Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 240-247). Stellenbosch, South Africa.

De Bock, D., Verschaffel, L., Janssens, D., & Rommelaere, R. (1999). What causes improper proportional reasoning: The problem or the problem formulation? *Proceedings of the 23rd Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 241-248). Haifa, Israel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht: Reidel.

Moyer, J.C., Sowder, L., Threadgill-Sowder, J., & Moyer, M.B. (1984). Story problem formats: Drawn versus verbal versus telegraphic. *Journal for Research in Mathematics Education, 15*, 342-351.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

Salomon, G. (1981). *Communication and education, social and psychological interactions.* Beverly Hills, CA: Sage Publications.

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process (Towards ... a theory). Part I: Reflections on a teaching experiment. *Educational Studies in Mathematics, 15*, 327-348.

Treffers, A. (1987). *Three dimensions. A model of goal and theory description in mathematics instruction.* The Wiskobas project. Dordrecht: Reidel.

Van Essen, G., & Hamaker, C. (1990). Using self-generated drawings to solve arithmetic word problems. *Journal of Educational Research, 83*(6), 301-312.

# THE DIFFICULTIES STUDENTS EXPERIENCE IN GENERATING DIAGRAMS FOR NOVEL PROBLEMS

Carmel M Diezmann
Centre for Mathematics and Science Education
Queensland University of Technology, Australia

**Abstract** *Although "draw a diagram" is advocated as a useful problem-solving strategy, generating an appropriate diagram is problematic for many students. This case study explored primary-aged students' difficulties in generating diagrams for novel problems. Three categories of difficulties were identified: (1) non-use of diagrams; (2) generic difficulties with diagrams and (3) idiosyncratic difficulties that were related to specific diagrams. The results suggest that there is a need for instruction in diagram use to empower students and address their difficulties. Specific attention needs to be given to: (1) the diagram-picture distinction; (2) the ambiguity of diagrams; and (3) the dynamic feature of diagrams.*

## Background

"For all a rhetorician's rules; Teach nothing but to name his tools." *Hudibras 1663*

The use of the strategy *draw a diagram* is strongly advocated as a tool for problem solving (e.g., National Council of Teachers of Mathematics, 1989). A diagram is a particularly effective problem representation because it exploits spatial layout in a meaningful way, enabling complex processes and structures to be represented holistically (Winn, 1987). Generating a diagram facilitates the conceptualization of the problem structure and is the first step towards a successful solution (van Essen & Hamaker, 1990). However, it is fallacious to assume that diagrams are spontaneously effective tools for students (e.g., Dufoir-Janvier, Bednarz, & Belanger, 1987). As inadequate diagrammatic representations limit students' problem-solving capabilities (Klahr 1978), it is important to investigate factors that influence problem representation (Goldman, 1986). The purpose of this paper is to explore students' difficulties in generating diagrams for novel problems.

Representing problem information on a diagram involves the decoding of linguistic information and the encoding of visual information. During this translation process, there is the potential for knowledge acquisition (Karmiloff-Smith, 1990) through the reorganisation of information (Weinstein & Mayer, 1986) and subsequent inference-making (Lindsay, 1995). Knowledge acquisition depends on three components, namely, selective encoding, selective combination and selective comparison (Sternberg, 1990). Selective encoding relates to the relevance of the information that is represented. Some students' representations are unhelpful for problem solving because relevant problem information is not included (Dufoir-Janvier et al., 1987). Selective combination refers to how new information is integrated as a discrete entity. The diagram is an effective problem representation because problem information is indexed by location on a plane, which supports a large number of inferences (Larkin & Simon, 1987). Selective comparison focuses on the relationship between new knowledge about the content of the problem or diagram and prior knowledge. These components highlight the importance of knowledge about the types of diagrams that have applicability in problem solving.

General purpose diagrams (i.e, networks, hierarchies, matrices, and part-whole diagrams) are particularly important in problem solving because they provide representational frameworks that are applicable to a range of problem structures. For example, a matrix can be used to represent the problem structure in combinatorial tasks or in deductive reasoning tasks. Novick (1996) developed a theoretical framework for spatially-oriented diagrams, namely, matrices, networks, and hierarchies, that describes the conditions of applicability and distinguishing properties for each of these diagram types. Her framework does not include part-whole diagrams, which are conceptually-oriented diagrams and have no unique external form.

## Design and Methods

The hypothesis that there would be an improvement in students' generation of diagrams after diagram-related instruction was tested using an explanatory case study design (Yin, 1994) and subsequently supported (Diezmann, 1999). As knowledge of students' difficulties is a key facet of pedagogical content knowledge (Carpenter, Fennema, & Franke, 1996), data from the case study was used in an inductive theory-building approach (Krathwohl, 1993) to develop a list of students' difficulties in generating a diagram to further inform instructional practice.

The participants in the case study were 12 Year 5 students with a mean age of 10 years 3 months from a moderately sized school in a lower socio-economic suburb in Brisbane, Australia. The participants comprised a cross-section of students, who were high and low performers in problem solving, and had high and low preferences for a visual method of solution. Isomorphic sets of five novel problems were presented to each participant, by an interviewer who was known to them, during 30 minute interviews conducted before and after instruction. The interviews were video-taped and subsequently transcribed. As the participants were not specifically instructed to use a diagram, those participants who did not spontaneously use a diagram were given further opportunities to generate a diagram.

## Selective Results

The students experienced three categories of difficulties generating effective diagrams: (1) non-use of a diagram; (2) generic difficulties; and (3) idiosyncratic difficulties. Examples of difficulties in each category follows. By necessity, examples are illustrative rather than comprehensive due to space limitations.

### Category 1: Non-use of a Diagram

The first category focuses on the reasons why students did not use a diagram. *Draw a diagram* is only one of many strategies that students might use in problem solving. However of particular concern are reasons why *draw a diagram* might not be part of students' repertoire of problem solving strategies.

*A lack of understanding of the mathematical use of the term "diagram":* Although the term "diagram" is commonly used in mathematics it cannot be assumed that it is understood by students as indicated by comments, such as "What's a diagram?" Further indication of a lack of understanding of the term can also be inferred from students' "diagrams" which depicted pictorial elements of a problem but lacked a representation of the relational elements of the problem (e.g., Figure 3). Confusion about the term is also demonstrated by students' synonymous use of the terms "diagram", "picture" and "drawing" without any qualification.

*A lack of understanding of the concept of a "diagram":* When students were unable to proceed with a non-diagrammatic strategy, they were encouraged to generate a diagram. However there was no recognition by some students that a diagram might assist them in problem solving. When the students were asked, "Would a diagram help?" their responses generally ranged from "no" to "maybe". Even when students responded positively, a reluctance to generate a diagram was still apparent. For example, while Frank conceded that a diagram might help, he baulked at drawing a diagram due to the time he perceived it would take. On a farmyard problem, he commented, "Yeh but it'd take ages because you'd have to draw a lot of chickens and a lot of pigs". Frank's comment suggests that he did not view the diagram as a means to the solution even when unsuccessful with another strategy.

*A lack of understanding of the diagram as a representation that utilises scale:* A feature of some diagrams is their facility to utilise scale, however a lack of knowledge of scale was one of the reasons why a particular student did not use a diagram. The following interaction between Jon (J) and the researcher (R) shows Jon's concern with depicting the size of the tree and his lack of knowledge of scale.

R:Is there anything that you can think of that you can draw that would help?
J: I thought of drawing a ten metre high, a ten metre tree and then each time going up five metres.
R:Have a quick go and see if you can do that, see if it's possible.
J: I don't have enough room to do a ten metre tree.
R:You don't have enough room for a ten metre tree?
J: (Shook his head.)

### Category 2: Generic Difficulties in the Generation of a Diagram

The second category provides examples of generic difficulties that students experienced in generating diagrams. When the same difficulty was identified for a range of general purpose diagrams it was also included in this category.

*Creating a diagram that was unusable:* The reasons why students self-generated diagrams were unusable included: (a) being too small to represent all the relevant information; (b) having insufficient space around the diagram to extend the diagram; and (c) being too untidy to clearly see the elements of the diagram. For example, untidiness was a problem for Candice (C) and led to her discarding her diagram (See Figure 1). Drawing an unusable diagram is particularly problematic because students generally abandoned the strategy rather than redrew their diagrams.

R: And I saw that you also had a diagram there but you scribbled it out. What happened?

C: It (the diagram) didn't work because it was really messy and I couldn't do it because um too complicated (pointed to the diagram).

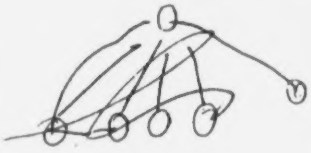
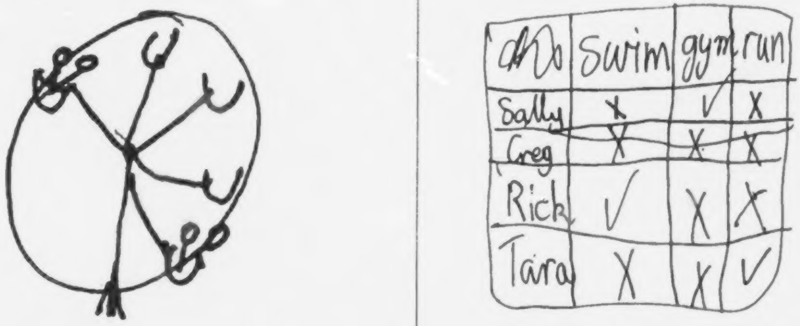


*Figure 1.* An untidy diagram.

*Incorrect representation of quantity:* A common error made by students was to represent quantities incorrectly. Examples of this error were evident in a variety of diagrams, such as those on Figure 2. While this error may simply reflect carelessness, it is an irrecoverable error because it was generally undetected by students and resulted in an incorrect answer. For example, in the matrix on Figure 2, Gemma did not detect her error even when she experienced difficulty utilising the clues in a deductive problem about a group of friends playing sport.



| who | swim | gym | run |
|---|---|---|---|
| Sally | X | ✓ | X |
| Greg | X | X | X |
| Rick | ✓ | X | X |
| Tara | X | X | ✓ |

"Six" people on a ride

Four people and "four" sports

*Figure 2.* Incorrect representations of quantity.

## Category 3: Idiosyncratic Difficulties in the Generation of a Diagram

The third category comprises the idiosyncratic difficulties that students experienced with networks, hierarchies, matrices, and part-whole diagrams. For illustrative purposes, examples of students' difficulties with networks and hierarchies are described for *The Koala* task and *The Party* respectively.

**The Koala**: A sleepy koala wants to climb to the top of a gum tree that is 10 metres high. Each day the koala climbs up 5 metres, but each night, while asleep, slides back 4 metres. At this rate, how many days will it take the koala to reach the top?

*A lack of precision in network diagrams:* Network diagrams are useful for representing location. However students' diagrams often lack the requisite information to ascertain precise locations. In *The Koala* task, the koala's location can be identified by the number of metres the koala is above the ground. The lack of metre marks on Helen's (H) diagram became problematic when she tried to ascertain the exact location of the koala at a certain point in time (See Figure 3).

H: … he had to he climbed up another five and then he slept again and the second day when he climbed up I mean the third day he climbed up to the top.

R: How do you know he climbed up to the top are you sure or could he have been just a bit lower?

H: He might have been a bit lower.

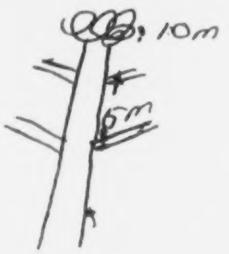


*Figure 3.* Difficulty identifying a precise location.

*Overlooking the constraints in a network diagram:* Locations on network diagrams are both provided in the problem information and produced as the student generates the diagram. In *The Koala* task, some students overlooked the height constraint of 10 metres as they progressively generated diagrams and tracked the koala's changing location. For example, Kate (K) produced the diagram in Figure 4 through a process of repeatedly moving up five metres and down four metres. However, she overlooked the goal of reaching 10 metres.

K: … and then he'd climb back up to number nine and slide back down to five and then he'd climb up to ten and slide back to six and then he'd climb up to eleven …

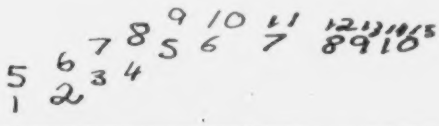


*Figure 4:* Extending beyond the constraints of the problem.

*Labelling the starting position incorrectly on the network diagram:* Measurement can also feature on network diagrams. A common error in *The Koala* task was to identify the base of the tree as one metre, as shown in the interaction between Damien (D) and the researcher (R). This error was not restricted to the use of standard measures but also occurred when students used non-standard measures. In another network problem, a number of students identified the base of a well as one brick high.

R: So where does he start?
D: He starts here. (Pointing to a line at the base of the tree)
R: He starts on that mark. What mark is that? What number?
D: The first one.
R: The first what?
D: Um. The first part of the tree.
R: Okay. If you told me in metres how many metres would it be?
D: One metre.

In summary, the difficulties students experienced with network diagrams tended to relate to location and measurement. Similarly, the difficulties students experienced with other general purpose diagrams were related to the uniqueness of each of those types of diagrams. Students' difficulties with matrices were associated with two-dimensional arrays; their difficulties with part-whole diagrams were related to determining the parts and wholes of sets; and their difficulties with hierarchies occurred in the representation of hierarchical relationships. An example of a difficulty unique to hierarchies follows.

**The Party:** At a party 5 people met for the first time. They all shook hands with each other once. How many handshakes were there altogether?

R: If you were telling me what to do how would I know which ones (lines representing people) to join?

G: Um. These two together (1-2 moving from left to right), those two together (2-3), those two together (3-4), and those two together (4-5) and those ones (Gemma retraced the lines from 5-4, 4-3, 3-2 but from right to left) and that (tracing over 1-5).

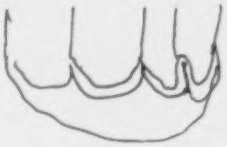


*Figure 5.* An incorrect relationship of a hierarchical situation.

*An incorrect relationship on a hierarchy:* Hierarchies represent information that either increases (e.g., a family tree) or decreases (e.g, a knockout tennis competition). In *The Party* task, there are progressively less people who need to shake hands so there is a hierarchical relationship among the number of handshakes that might be initiated by each individual. While Gemma (G) has some understanding that people need to shake hands with one another, the hierarchical relationship is neither apparent in her diagram nor in her explanation (See Figure 5).

## Discussion and Conclusions

Students' difficulties in generating a diagram indicate that, despite its potential, the strategy *draw a diagram* was initially not an effective problem-solving tool for many students. While explanations for students' non-use of diagrams, and their generic and idiosyncratic difficulties differed from student to student, their difficulties were essentially related to a lack of knowledge about the affordances and constraints of diagrams as tools for problem solving. Clearly, to be empowered, students need to be educated in the use of the diagram as a problem-solving tool. Students need to know why a diagram can be useful in problem solving, which diagram is appropriate for a given situation, and how to use a diagram to solve a problem. In addition, to this explicit content, there is a need to address three further issues related to diagram use.

First, students need to distinguish diagrams from other pictorial representations and understand their relative purposes. There are substantive differences between diagrams and pictures or drawings. Surface details are generally important in a picture, while structural features are important in a diagram (Dufoir-Janvier et al, 1987). Additionally, a picture is a static knowledge-representation system — a snapshot — while a diagram is a knowledge-generating system that is designed to support inference making (Lindsay, 1995). Using the terms "diagram", "picture" and "drawing" synonymously fails to distinguish diagrams from other pictorial representations and may lead to confusion.

Second, students need to understand the nuances of ambiguity associated with diagrams. Diagrams by their nature are an ambiguous representation and problem information can be variously depicted. However, what is important is that the arrangement of the information on the diagram represents the structure of the problem. While some representation of the surface features can be helpful as a "reminder" about particular elements on the diagram, a focus on representing surface features is counterproductive if the student is distracted from representing the problem structure.

Third, students need to develop awareness that diagrams are dynamic rather than static representations. Diagrams are "physical" working spaces for trialing relationships among the elements of the problem, and, hence, need to be sufficiently large and relatively neat. As an understanding of the problem can evolve through the generation of a diagram (Nunokawa, 1994), it can be beneficial to produce more than one diagram.

Diagrams are an important tool for problem solving, however the benefits of any tool are closely associated with the users' knowledge of the tool, their opportunities to observe master craftspersons, and the development of their skill in using the tool. Advocating that students *draw a diagram* without addressing their difficulties and educating them about diagrams is quite simply the waste of a very good tool!

## References

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20.

Diezmann, C. M. (1999). *The effect of instruction on children's use of diagrams in novel problem solving.* Unpublished PhD thesis, Queensland University of Technology, Australia.

Dufoir-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Representation in the teaching and learning of mathematics.* (pp. 109-122). New Jersey: Lawrence Erlbaum.

Goldman, A. I. (1986). *Epistemology and cognition.* Cambridge, MA: Harvard University Press.

Karmiloff-Smith, A. (1990). Constraints on representational change: evidence from children's drawing. *Cognition, 34*, 57-83.

Klahr, D. (1978). Goal formation, planning, and learning by pre-school problems solvers, or: My socks are in the dryer. In R.S. Siegler (Ed.), *Children's thinking: What develops?* (pp.181-212). Hillsdale, NJ: Erlbaum.

Krathwohl, D. R. (1993). *Methods of educational research.: An integrated approach.* White Plains, NY: Longman.

Larkin, J. H. & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11*, 65-99.

Lindsay, R. K. (1995). Images and inferences. In J. Glasgow, N. H. Narayanan, & B. C. Karan. *Diagrammatic reasoning* (pp. 111-135). Menlo Park, CA: AAI Press.

National Council of Teachers of Mathematics. (1989). *Standards.* Reston, VA: Author.

Novick, L. R. (1996, April). *On the nature of college students' knowledge about spatial diagram representations.* Paper presented at the Third International Conference on Thinking, London.

Nunokawa, K. (1994). Improving diagrams gradually: One approach to using diagrams in problem solving. *For the Learning of Mathematics, 14*(1), 34-38.

Sternberg, R.J. (1990). *Metaphors of mind.* Cambridge, UK: Cambridge Uni. Press.

van Essen, G., & Hamaker, C. (1990). Using self-generated drawings to solve arithmetic word problems. *Journal of Educational Research, 83*(6), 301-312.

Weinstein, C. E., & Mayer, R. (1986). The teaching of learning strategies. In M. Wittrock (Ed.), *Handbook of research on teaching* (pp. 315-327). New York, NY: McMillan.

Winn, B. (1987). Charts, graphs, and diagrams in educational materials. In D.M. Willows & H.A.Houghton (Eds.). *The psychology of illustration* (Vol 1, pp. 152-198). New York, NY: Springer-Verlag.

Yin, R. K. (1994). *Case study research: Design and methods* (2nd ed.). Thousand Oaks, CA: Sage.

# ABOUT ARGUMENTATION AND CONCEPTUALISATION

Nadia DOUEK
I.U.F.M. de Creteil
France

Ezio SCALI
Scuola Elementare di Piossasco
I.M.A. del C.N.R, Genova - Italia

**ABSTRACT**: *this report presents an analysis of different functions of argumentation in the construction of mathematics concepts in primary school. Based on the definition of "concept" by G. Vergnaud and the elaboration about "scientific" concepts by L. S. Vygotsky, we will consider how argumentation intervenes in the progressive construction of basic mathematics concepts and in the development of consciousness and systemic links between concepts. A sequence of five activities on the same problem in a second grade class will be considered in order to provide experimental evidence for some aspects of this analysis.*

## 1. Introduction

Vergnaud (1990) defined a concept as the system consisting of three components: the reference situations, the operational invariants ('schemes', theorems in actions, etc.) and the symbolic representations. This definition can be very useful in school practice, because it allows teachers to follow the process of conceptualisation in the classroom by monitoring students' development of the three components. From the research point of view, Vergnaud's definition raises some interesting questions - in particular, questions about the progressive constitution of the three components: how does an experienced situation become a "reference situation" for a given concept? What are the relationships between the acquisition of the symbolic representations of a given concept and the construction of its operational invariants?

In the Vygotskian elaboration about "scientific" concepts (see Vygotsky, 1985), consciousness, intentionality, and developing concepts into systems are considered as crucial features of "scientific" concepts. Vygotsky's elaboration suggests other questions: how consciousness about symbolic representations and operational invariants of a given concept can be attained (as a condition for their appropriate, intentional use in problem solving)? How concepts can be developed into systems1)?

As we will see in the next sections, the study of argumentation in relationship with conceptualisation can play an important role in tackling the abovementioned questions.

## 2. Argumentation

Research about argumentation has strongly developed in the last four decades. Different theoretical frameworks have been developed, based on different research perspectives: from the analysis of the pragmatics of argumentation - *"argumentation to convince"*, Toulmin (1958); to the analysis of the syntactic aspects of argumentative discourse - *"polyphony of linguistic connectives"*, in Ducrot (1980). A few studies have considered the specific, argumentative features of mathematical activities; we may quote the cognitive analysis of argumentation vs proof by Duval (1991) and some more recent studies about the interactive constitution of argumentation (Krummeheuer, 1995) and interactive argumentation in explanation and justification (Yackel, 1998). In our opinion, if we want to consider the role of argumentation in conceptualisation, we need to reconsider what argumentation can be in mathematical activities, focusing also on its "logical" structure and use of arguments belonging to "reference knowledge" (Douek, 199b) and not only on its syntactic aspects or its functions in social interaction.

We shall use the word "argumentation" both for the process which produces a logically connected discourse about a subject (Webster dictionary: "*1 - the act of forming reasons, making inductions, drawing conclusions, and applying them to the case under discussion*"), as well as the text produced by that process (Webster: "*3 - writing or speaking that argues*"). The discourse context will suggest the appropriate meaning. "Argument" will be "*a reason or reasons offered for or against a proposition, opinion or measure*" (Webster); it may include linguistic arguments, numerical data, drawings, etc. So, an "argumentation" consists of some logically connected "arguments". If considered from this point of view, argumentation plays crucial roles in mathematical activities: it intervenes in conjecturing and proving as a substantial component of the production processes (see Douek, 1999a); it has a crucial role in the construction of basic concepts during the development of geometric modeling activities (see Douek, 1998; 1999b).

## 3. Argumentation and Conceptualisation

With reference to Vergnaud's and Vygotsky's elaborations[1)] about concepts, we will consider conceptualisation as the complex process which consists in the construction of the components of concepts considered as systems[1)], in the construction of the links[1)] between different concepts and in the development of consciousness about them. The main issue of this report is to analyse the possible functions of argumentation in conceptualisation.

### *3.1. Experiences, Reference Experiences and Argumentation*

An experience can be considered as a reference experience for a given concept when it is referred to as an argument to explain, justify or contrast in an argumentation concerning that concept[2)]. The criterion applies both to basic experiences related to elementary concept construction and to high-level, formal and abstract experiences. As a consequence of our criterion, to become a reference experience for a given concept, an experience must be connected to symbolic representations of that concept in a conscious way (in order to become an argument intentionally used in an argumentation). By this way, a necessary functional link must be established between the constitution of reference experiences for a given concept and its symbolic representations. Argumentation may be the way by which this link is established (see Section 4 for examples; see also Douek, 1998; 1999b).

The subject develops argumentation skills and constitutes reference experiences for concepts through a dialectical process. Argumentation can be seen as a means to develop an experience into a reference experience for a given concept through two connected ways: it involves the subject's view and consciousness about that concept in the experience; and it involves some symbolic representations of that concept in the experience. These two ways create semantic roots for the representations relating the experience to the network of the subject's conscious knowledge. On the other hand, one clearly needs reference experiences as arguments and backings in an argumentative process concerning a given concept.

### *3.2. Argumentation and Operational Invariants*

Argumentation allows to make explicit operational invariants and ensure their conscious use. This function of argumentation strongly depends on teacher's mediation and is fulfilled when students are asked to describe efficient procedures and the conditions of their appropriate use in problem solving. The comparison between alternative procedures to solve a given problem can be an important manner of developing consciousness (see 4.2.). The inner nature of concept as a "system" is enhanced through these argumentative activities: different operational invariants can be compared and connected with each other and with appropriate symbolic representations, thus revealing important aspects of the system.

### *3.4. Argumentation, Discrimination and Linking of Concepts*

Argumentation can ensure both the necessary discrimination of concepts and systemic links between them. These two functions are dialectically connected: argumentation allows to separate operational invariants and symbolic representations between "near" concepts (for an example see Douek, 1998 and 1999b: argumentation about the expression *"height of the Sun"* allows to distinguish between *"height"* to be measured with a ruler, and *"angular height of the Sun"* to be measured with a protractor). And by the same way possible links between "near" concepts can be established.

## 4. Examples

### *4.1. A General Description of the Educational Environment*

The examples come from a second grade class, a participant in the Genoa Project for Primary School. The aim of this project is to teach mathematics, as well as other important subjects (native language, natural sciences, history, etc.), through systematic activities concerning "fields of experience" from everyday life (Boero et al., 1995). For instance, in Grade I the "money" and "class history" fields of experience ground the development of numerical knowledge and initiate argumentative skills as well as the use of specific symbolic representations.

A fairly common classroom routine consists of: individual production of written hypotheses on a given task; classroom comparison and discussion of student products guided by the teacher; individual written reports about the discussion; and classroom summary, usually constructed under the guidance of the teacher and written down by the students in their exercise books. In most cases, classroom summaries represent the knowledge acquired by the students (with all possible ambiguities and hidden mistakes). It is only in a few circumstances that the final institutionalisation phase (Brousseau, 1986) is reached. This style of slowly evolving knowledge without "sure" and final "truth" offers the opportunity to observe the transformation of the students' knowledge in a favourable climate: such transformations are normal, expected events. Moreover, argumentation reflects fairly well each student's level of mastery of knowledge, his/her level of use of references, and the common stable references for the class.

In all Italian primary schools, the student group stays with the same two or three teachers for the whole five years of primary education. Therefore didactical contracts are very stable, and once established, maintaining them is effortless.

We will consider data deriving from direct observation, the students' texts, and videos of classroom discussions in a second grade class of twenty students.

### *4. 2. Measuring the Height of Plants in a Pot with a Ruler*

We shall analyse a classroom sequence consisting in five activities concerning the same problem: students had to measure wheat plants (that they grew in the classroom) in their pot, using their rulers. Students had already measured wheat plants taken out of the ground in a field, now they had to follow the increase in time of the heights of plants of the classroom pot. The difficulty was that the rulers usually do not have the zero mark at the edge, and it was not allowed to push the ruler into the ground (in order to avoid harming the roots). Children had to find a general solution (not concerning a specific plant). In particular, they could use either the idea of translating the numbers written on the rulers (by using the invariability of measure through translation) - an act we call the "translation solution"- or the idea of reading the number at the end of the plant and then adding to this result the measure of the length between the edge of the ruler and the zero mark (by using the additivity of measures) – an act we call the "additive solution".

The first activity was a one-to-one discussion with the teacher to find out how to measure the plants in the pot. The available ruler had a space of an apparent one centimeter length between the zero mark and its edge. The purpose of this discussion was to arrive at solutions that might be reached and declared by the student at the end of the interaction.

The main difficulties met by students were:

- Focus on the problem: There were many plants in the pot; some students concentrated on twisted plants; one worried about not harming the roots; another insisted on the idea of using a professional tape measure (with the zero mark at the edge of the tool). The teacher, T, used argumentation (in interaction with the student, S) in order to focus on the problem, like in the following excerpt:

(with the help of the teacher, this student has already discovered that the number read on the ruler that corresponds to the edge of the plant is not the measure of the height of the plant)

S: *We could pull the plant out of the ground, as we did with the plants in the field*
T: *By this way we cannot follow the increase of height of our plants*
S: *We could put the ruler into the ground in order to bring zero to the level of the ground*
T: *But if you put the ruler into the ground, you could harm the roots*
S: *I could break the ruler, removing the piece under zero*
T: *It is not easy to break the ruler exactly on the zero mark; then the ruler is damaged.*

- Once focused on the problem, there remains the question of how to go beyond the discovery that the measure read on the ruler is not the height of the plant. In particular, it was difficult for some students to imagine either shifting the scale on the ruler in order to bring zero at the edge of the ruler or measuring the length between zero and the edge of the ruler. Concerning the former difficulty, usual classroom practices on the "line of numbers" (shifting numbers or displacing them by addition) had to be transferred to the new situation of measuring. In general, a change of the status of the ruler was needed: from a tool to get a measure, to an object to be measured or transformed (e.g., cut or bent). With the help of the teacher, most students overcame difficulties by different ways. In particular, some of them imagined putting the ruler into the ground in order to bring the zero mark to the level of the ground, but since this action could harm the roots, they imagined shifting the number scale along the ruler.

S: *I could put the ruler into the ground*

T: *If you put the ruler into the ground you could harm the roots*

S: *I must keep the ruler over the ground... but then I can imagine bringing zero below, on the ground, and then bring one to zero, and two to one ... It is like if the numbers slide downwards.*

Other students imagined cutting the ruler. When they were not allowed to do so, they imagined making a piece of the ruler (or the plant) and bringing it to the top of the plant.

S: *I would like to cut this piece of the ruler*

T: *Then you damage the ruler*

S: *But I can imagine to cut this piece without doing it... I can also imagine to cut the plant and bring this piece to the top of the plant, where I can measure it...*

During this phase, reference was made (in many cases by the student, in some cases by the teacher) to preceding experiences of measuring in simple situations or activities on the number line.

- Other difficulties were: to keep the line of reasoning during the interaction; and to reconstruct a whole reasoning (built in interaction with the teacher) at the end of the interaction and then dictate the resulting procedure to the teacher.

At the end of the interaction, 9 students out of 20 arrived to a complete solution (i.e. were able to dictate an appropriate procedure): 4 were "translation" solutions, 4 "additive" solutions and 1 a mixed solution (with an explicit indication of the two possibilities, "addition" and "translation").

Rita's "translation" solution: *In order to measure the plant we could imagine that the numbers slide along the ruler, that is zero goes to the edge, one goes where zero was, two goes where one was, and so on. When I read the measure of the plant I must remember that the numbers have slid: if the ruler gives 20 cm, I must consider the number coming after 20, namely 21.*

Alessia's "additive" solution: *We put the ruler where the plant is and read the number on the ruler, which corresponds to the height of the plant, and then add a small piece, that is the piece between the edge of the ruler and zero. But before we must measure that piece.*

4 students moved towards a "translation" solution without being able to make it explicit at the end of the interaction. The other 7 students reached only the consciousness of the fact that the measure read on the ruler was the measure of one part of the plant, and that there was a "missing part", without being able to establish how to go on.

The second activity consisted in an individual written production. The teacher presented a photocopy of Rita's and Alessia's solutions, asking the students to say who's solution was like theirs, and why. This task was intended to provide all students with an idea about the solutions produced in the classroom. With one exception, all the 13 students who had produced or approached a solution were able to recognize their solution or the kind of reasoning they had started. And 6 out of the other 7 students declared that their reasonings were different from those produced by Rita and Alessia.

The third activity was a classroom discussion. The teacher at the blackboard, and the students working on their exercise books (where they had drawn a pot with a plant in it) and using a paper ruler similar to that of the teacher, effectively put into practice the two proposed solutions, first the "translation" solution and then the "additive" solution. The "arrow" representation of addition was spontaneously used by some students to represent either the shifting of the number scale along the ruler or the transfer of the bottom piece of the ruler (or the plant) to the top of the plant. The teacher discussed this matter with the students. Meanwhile students discussed some problematic points that emerged. In particular, they discussed the fact that while the "translation" procedure was easy to perform only in the case of a length (between the zero mark and the edge of the ruler) of 1 cm (or eventually 2 cm), the "additive" solution consisted in a method easy to use in every case. Another issue they discussed concerned the interpretation of the equivalence of the results provided by the two solutions ("*why do we get the same results?*").

Here is an excerpt from the discussion, concerning the starting point of the comparison of the two solutions:

Angelo: *Rita's method is similar to Alessia's method*

(many voices: *No, no...*)

Ilaria: *Rita makes the numbers slide, on the contrary Alessia... she does not make it... The two methods are not similar*

Jesssica: *Rita says to make the numbers slide, but Alessia moves the piece of the plant*

Teacher: *Wait a moment, please. Jessica probably has catched an important point. She says:* 'Rita makes the numbers slide'... *the measure of the plant for Rita is always the same, they are the numbers that slide... While, as Jessica says, Alessia has imagined to take a piece of the plant and to bring it at the edge of the plant, where we can measure it..*

Angelo: *I do not say that it is the same thing; I say that the two methods are a little bit similar*

Giulia: *It is like if Alessia would overturn the plant,... she would bring a piece over... a small piece went over and the plant seems to be hanging on... so we could measure it..*

The fourth activity was an individual written production where students had to "*explain why Rita's method works, and explain why Alessia's method works*".

With three exceptions (who remained rather far from a clear presentation, although they showed an "operational" mastery of the procedures), all the students were able to produce the explanations demanded by the task. One half of the students added comments about the two methods; most of them explained in clear terms the limitation of the "translation" solution. Here is an example:

Marco: *Rita's reasoning works, because it makes the ruler like a tape measure, because zero goes at the edge of the ruler. She imagined the same thing and made the numbers slide. Where I see that the edge of the plant is at 8 cm, she says 'to slide' and sees that the ruler slid by one centimeter, and so she sees that 8 became 9. But this method works easily only if the ruler has a space of 1 cm between zero and the edge.*

*Alessia's reasoning works because Alessia adds the piece of plant she carries to the height of the plant and adds 1 to 10 and she sees that it makes 11.*

*One difference consists in the fact that Rita leaves the piece of the plant where it is while Alessia carries it up to 10.*

The fifth activity consisted in the classroom construction of a synthesis to be copied on students' exercise books. It was not difficult to get an exhaustive synthesis, due to the good quality of many students' individual productions.

In the following classroom activities, the "*measure of the plants in the pot with the ruler*" was recalled as a reference when students had to measure the length of objects not accessible in a direct reading of their length on the ruler.

### *4.3. Some Comments about the Available Data*

The available data show experimental evidence for some of the potentialities of argumentation as stated in the preceding sections. In particular:

- during the interactive resolution of the problem (first activity), the argumentative activity with the teacher brought students to grasp the nature of the problem and transformed the situation into a possible "reference experience" for the involved operational invariants of the measure concept. Teacher's arguments compelled students to move from imagining physical actions (putting the ruler into the ground or cutting the ruler) to operations involving operational invariants of the measure concept (translation invariance or additivity of measure of length). The argumentative interaction with the teacher involved many arguments (together with their symbolic representation, such as gestures, linguistic expressions, etc.) which originated from preceding activities of playing on the number line or measuring. These arguments were related to the new experience and prepared the ground for personal constructions and descriptions of appropriate procedures to solve the problem implicitly based on the abovementioned operational invariants.
- During the classroom discussion (third activity), argumentation fulfiled different functions: it allowed to make explicit the two different operational invariants ("translation invariance" and "additivity") of the measure of lengths and the systemic links between them and with other concepts (addition and subtraction) in the "conceptual field" of the "additive structures" (Vergnaud, 1990). It also allowed to transfer to the new situation some symbolic representations, like the "arrow" representation of addition and subtraction, used in other contexts (the "money and purchase" field of experience, or activities on the number line).
- During the last individual activity ("*explain why...*") argumentation established personal (Leont'ev, 1978) links between the two methods, being a way of revealing and enhancing the internalisation (Vygotsky; see Wertsch, 1991) of the acquisitions constructed (at the interpersonal level) during the preceding discussions. By this way, students attained a first level of consciousness about potential and limitations of the operational invariants involved.

## 5. Conclusions and Educational Implications

Our report presents (and illustrates with a few examples) some important functions that argumentation (as a logical, but not necessarily deductive, discourse) can fulfill in conceptualisation. The examples make evident the fact that these functions of argumentation represent a potential that needs careful teacher's mediation in order to be exploited. Teacher's mediation must consist in:

- appropriate tasks, in order to focus on the crucial points of conceptualisation;
- appropriate argumentations during the 1-1 interactions with students, in order to focus on the problem and transform the problem situation into a reference situation for some operational invariants of the measure concept;
- the choice of suitable students' productions to be compared and discussed in the classroom, in order to prepare the ground for the following activities;
- the management of the discussions, in order to bring important points to surface and make them explicit.

The actors of argumentation on the scene can be different: argumentation can intervene in the teacher-student 1-1 interactions, as a component of students' discourses in classroom discussions, and as an individual student's production. Each kind of argumentation fulfills specific functions, as seen in the preceding section.

## NOTES

1) Although the two theoretical frameworks are different, they seem to be locally compatible in the specific issues that are dealt with in this report. In particular, both authors consider a double side systemic character of concepts: concepts as systems, and systemic links between "near" concepts. But we must remember that in the case of "scientific" concepts (Vygotky) these systemic links come from the cultural mediation of the teacher; while "conceptual fields" (Vergnaud) are cultural systems based on structural links and mainly constructed through adaptation.

2) We can imagine that if the experience is referred to in the child's processes of thinking, then it should also be a reference experience. But this is not quite certain from a theoretical point of view: what is the stage of differentiation of knowledge when it is not put into language?

## REFERENCES

Boero, P.; Dapueto, C.; Ferrero, E.; Garuti, R.; Lemut, E.; Parenti, L.; Scali, E.: 1995, 'Aspects of the Mathematics-Culture Relationship', *Proc. of PME-XIX*, Recife, vol. 1, pp. 151-166.

Brousseau, G.: 1986, 'Fondements et méthodes de la didactique des mathématiques', *Recherches en didactique des mathématiques*, 7, 33-115.

Douek, N.: 1998, 'Analysis of a Long Term Construction of the Angle Concept in the Field of Experience of Sunshadows', *Proc. of PME-XXII*, Stellenbosch, vol. 2, pp. 264-271.

Douek, N.: 1999a, 'Argumentative aspects of proving: analysis of some undergraduate mathematics students' performances', *Proc. of PME-XXIII*, Haifa, vol. 2, pp. 273-280.

Douek, N.: 1999b, 'Argumentation and conceptualisation in context: a case study on sun shadows in primary school', *Educational Studies in Mathematics* (in press).

Ducrot, O.: 1980, *Les échelles argumentatives*, Ed. de Minuit, Paris.

Duval, R.: 1991, 'Structure du raisonnement déductif et apprentissage de la démonstration', *Educational Studies in Mathematics*, 22, 233-261.

Krummheuer, G.: 1995, 'The ethnography of argumentation', in P. Cobb & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning*, pp. 229-269, L.E.A., Hillsdale, NJ.

Leont'ev, A. N.: 1978, *Activity, Consciousness and Personality*, Prentice-Hall, Englewood Cliffs.

Toulmin, S.: 1958, *The Uses of Argument*, Cambridge University Press, Cambridge.

Vergnaud, G.: 1990, 'La théorie des champs conceptuels', *Recherches en Didactique des Mathématiques*, 10, 133-170.

Vygotsky, L. S.: 1985, *Pensée et langage*, Editions Sociales, Paris.

Wertsch, J. V.: 1991, *Voices of the Mind*, Wheatsheaf, Harvester.

Yackel, E.: 1998, 'A Study of Argumentation in a Second-Grade Mathematics Classroom', *Proc. of PME-XXII*, Stellenbosch, vol. 4, pp. 209-216.

# GESTURE AND ORAL COMPUTATION AS RESOURCES IN THE EARLY LEARNING OF MATHEMATICS

Jan Draisma

Universidade Pedagógica, Beira – Mozambique

*This report presents the first results of an experimental programme of gesture and oral computation, designed for the improvement of early mathematics teaching and learning. The report refers to a pilot study, carried out in a grade 1 class of 74 pupils, in a semi-rural primary school 15 km from Beira's city centre. Gestures were used to support five-structured and ten-structured computation methods as an alternative for less productive counting strategies. All computation was verbalized in the four languages spoken by the pupils: in Portuguese, as language of instruction and in the local Ndau, Chuwabo and Sena languages. Modified number words based on ten and five were introduced in Portuguese, Ndau and Sena, in order to give them the same regular structures as they have in Chuwabo and have these correspond directly to gesture computation.*

## 1. INTRODUCTION

There are rather big divergences between the initial Mathematics taught in different countries. In the United States of America, *counting strategies* are taught and practiced during the first years (Baroody 1987) whereas in countries like Japan, China, Taiwan, South Korea, Germany and The Netherlands, counting strategies are considered preschool activities and the school teaches already more advanced computation strategies. In Germany and The Netherlands there is a long tradition of *oral computation*, called "computation in the head". In Japan and The Netherlands the use of the *auxiliary base five* is explored, in addition to the general base ten structure of spoken numeration and computation. Specific didactic devices are used by the pupils in school, corresponding to a 1–5–10 structure: Tiles and the traditional abacus (*soroban*) in Japan (Hatano 1982), the computing frame with 2 rows of 10 counters and strings of beads, painted in groups of 5, in The Netherlands (Treffers et al. 1990).

In China, Japan, Taiwan and South Korea recomposition methods structured around ten are taught in grade one (Fuson & Kwon 1992). These are the same methods as are traditionally taught in Germany and The Netherlands. Recent developments in these two countries indicate that now a less modest place is reserved for counting strategies (Padberg 1986, Treffers et al. 1990).

## 2. CULTURAL SUPPORT FOR COMPUTATION STRATEGIES

Fuson & Kwon (1992, p. 159) stress the importance of cultural support for counting and computation strategies. They notice that in Korea, the spoken numerals in combination with specific systems of finger counting support the use of the ten-structured methods of computing sums and differences to 18. They indicate three reasons, why "children in the United States lack the cultural support for ten-structured methods identified above for Korean children": (1) textbooks do not

266

support the ten-structured methods or the prerequisites for them; (2) the English words do not name the ten and the ones in numbers between 10 and 20. Therefore this ten is not available in a sum to suggest a ten-structured method; (3) there is not a culturally supported way to reuse fingers to show numbers larger than ten.

Hatano is conscious that "the conceptual model for addition and subtraction and the corresponding method of teaching early mathematics are closely associated with the Japanese culture", through the regularity and simplicity of the number words, through the use of the abacus and the auxiliary base five, and through the Japanese cultural emphasis on computational competence. He concludes that the Japanese experience "may not be applicable to other cultures without considerable modifications".

Although the very regular spoken numeration in Japanese, Chinese and Korean is seen as an important advantage in early arithmetic, especially for the ten-structured methods, the irregular spoken numeration in German and Dutch has not hindered the development of a strong tradition of oral computation, based on these methods.

## 3. EARLY MATHEMATICS IN MOZAMBIQUE

Recent studies in Mozambique show that in the early primary grades counting strategies predominate, often the most elementary including toe counting, although the syllabus and teacher's guides recommend strategies of verbal computation and ten-structured methods (Kilborn 1990, Draisma 1999, INDE 1989). In general, pupils use sticks or pebbles, or they resort to the use of strokes in their exercise books. Many teachers show a certain resistance against finger counting, probably as a result of schoolmaterials, which stress the use of sticks, pebbles and leaves and never speak about the possibility of using hands. In this respect, Portuguese materials used before Independence (1975) and Mozambican materials produced after Independence are similar. On the other hand, many teachers have a strong tendency to divert their pupils' attention towards written procedures, even before children have mastered a good proportion of the basic facts of addition and subtraction. (Draisma 1999).

Interviews with unschooled adults in Mozambique, however, show well developed and varied skills in *verbal computation*, similar to those advocated in Germany, The Netherlands, China, Japan, South Korea and Taiwan (Draisma 1998). These interviews were conducted in ten different Mozambican languages. The main characteristics of the numeration in these languages are:

a) Regular systems based on *ten*: *eleven* is said as *ten-and-one*, *twelve* as *ten-and-two*, *twenty* as *two-tens*, etc.

b) The majority of the languages use also the auxiliary base *five*, i.e., *six* is called *five-and-one*, *seven* is *five-and-two*, and, for instance, *seventy-four* is called *five-and-two-tens, and four*.

For the interviewers, all experienced adult educators or primary school teachers, it was an importantant discovery to see that unschooled adults were able to calculate verbally, without knowing how to write numbers and without resorting to counting.

## 4. CULTURAL SUPPORTS FOR COMPUTING STRATEGIES IN MOZAMBIQUE

In Mozambique there are several cultural factors which may support an early mathematics programme in which computation instead of counting is stimulated:

a) All Mozambican languages have regular number words structured on ten.
b) The majority of these languages use also the auxiliary base five.
c) Well developed skills of oral computation are found all over the country amongst non-schooled adults.
d) Gesture numeration is part of the Mozambican cultures (Gerdes & Cherinda 1993). Finger counting and gestures are the natural and historical equivalents of the 1–5–10 structured manipulatives utilized in some other countries.

A programme which explores these factors could contribute to an important improvement of early Mathematics learning in Mozambique.

## 5. THE EXPERIMENTAL PROGRAMME

In 1999 an experimental programme was designed for the use in Grade 1 in primary schools in the Beira region, with the objective of improving Mathematics teaching and learning through the exploration of:

a) verbal computation in Portuguese – the language of instruction in schools;
b) verbal computation in local Mozambican languages (Ndau, Sena and Chuwabo);
c) the use of *gestures and subitizing* instead of *finger counting* as a way of visualizing and feeling the computation in bases five and ten.
d) a modified numeration in Portuguese and the other languages which lack the base 5, in order to support ten-structured and five-structured computing strategies through explicit number words:

| # | Portuguese | Portuguese - 5 | Sena | Sena - 5 | Chuwabo |
|---|---|---|---|---|---|
| 1 | um | um | cibodzi | cibodzi | modha |
| 2 | dois | dois | piwiri | piwiri | bili |
| 3 | três | três | pitatu | pitatu | tharu |
| 4 | quatro | quatro | pinai | pinai | nai |
| 5 | cinco | cinco | pixanu | pixanu | tanu |
| 6 | seis | cinco-e-um | pitanthatu | pixanu na cibodzi | tanu na modha |
| 7 | sete | cinco-e-dois | pinomwe | pixanu na piwiri | tanu na bili |
| 8 | oito | cinco-e-três | pisere | pixanu na pitatu | tanu na tharu |
| 9 | nove | cinco-e-quatro | pipfemba | pixanu na pinai | tanu na nai |
| 10 | dez | cinco-e-cinco | khumi | pixanu na pixanu | tanu na tanu |
| 11 | onze | dez-e-um | khumi na cibodzi | khumi na cibodzi | kumi na modha |
| 12 | doze | dez-e-dois | khumi na piwiri | khumi na piwiri | kumi na bili |
| 13 | treze | dez-e-três | khumi na pitatu | khumi na pitatu | kumi na tharu |
| 14 | catorze | dez-e-quatro | khumi na pinai | khumi na pinai | kumi na nai |
| 15 | quinze | dez-e-cinco | khumi na pixanu | khumi na pixanu | kumi na tanu |

| | | | | | |
|---|---|---|---|---|---|
| 16 | dezasseis | dez-e-seis | khumi na pitanthatu | khumi na pixanu na cibodzi | kumi na tanamodha |
| 17 | dezassete | dez-e-sete | khumi na pinomwe | khumi na pixanu na piwiri | kumi na tanabili |
| 18 | dezoito | dez-e-oito | khumi na pisere | khumi na pixanu na pitatu | kumi na tanatharu |
| 19 | dezanove | dez-e-nove | khumi na pipfemba | khumi na pixanu na pinai | kumi na tananai |
| 20 | vinte | dez-e-dez,<br>dois dezes | khumi na khumi,<br>makumi mawiri | khumi na khumi,<br>makumi mawiri | kumi na kumi,<br>makumi meli |

The experimental programme may be seen as a specific interpretation of the existing national syllabus. The programme will be given by "ordinary" teachers, who show interest and who will be trained and accompanied by the researcher. In this way the project intends to obtain information not only about the pupils' learning but also about the handling and understanding of the programme by "ordinary" teachers.

**Research questions**

1. How do teachers and pupils react in relation to the use of the local languages in general and the mother-tongue in particular?
2. How do teachers and pupils react to the use of gesture computation, instead of counting strategies supported by sticks or pebbels?
3. How do teachers and pupils react to the change in perspective: from written registration of the computation towards the meaningful verbalization of computations?

## 6. THE PILOT STUDY

A pilot study was held in a primary school, located 15 km from Beira's city centre, in a semi-rural/semi-urban setting. This study, conducted during the months of September and October 1999, was carried out in a class of 74 pupils of Grade 1. The experimental lessons were given by a primary school teacher with whom the researcher had worked in previous years (hereafter to be called "the teacher"). All lessons were observed by and discussed with the class teacher, who also gave some of the experimental lessons ("the class teacher" is responsable for all other lessons). During two months, gesture calculation up to 20 was accompanied by verbalization in the *four main languages* represented in the class: Portuguese (mothertongue of 12% of the pupils), Ndau (48%), Chuwabo (25%) and Sena (12%). Of these languages, only Chuwabo uses the auxiliary base five. A meeting was held with the parents of the pupils, in order to explain that in the Mathematics lessons the various local languages would be used apart from Portuguese, the language of instruction. The parents supported this initiative of using the local languages.

## 7. RESULTS OF THE PILOT STUDY

### The class teacher

The class teacher explained that she always requires the pupils to have a collection of sticks with them for use in the Maths lessons. She never encourages the use of finger counting. She agreed with the experiment of gesture computation. After a few lessons, she concluded that the gestures were much easier to use than the sticks.

The class teacher, whose mother-tongue is Ndau but who speaks Portuguese at home with her children, concluded that computation in Chuwabo was easier than in her own language, because in Chuwabo there are less numerals to learn, and in Ndau there are not only more numerals, but some of them are difficult words, like *zvirongomuna* (four), *zvitanthatu* (six), *zvinomwe* (seven), and *zvipfumbamwe* (nine). She noted the pupils' difficulties with these numerals, also among the Ndau speaking pupils.

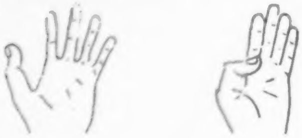
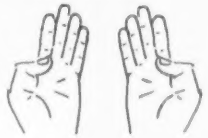
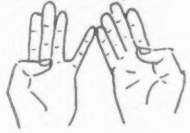
The class teacher contacted old people and shopkeepers, in order to be sure of the correct numerals in Ndau and Sena. She contributed with several corrections to the materials prepared by the researcher and the lists of numerals contained in Gerdes' *A numeração em Moçambique*. In particular, she and her colleague helped clarify the difference between ordinal numbers (used for counting purposes) and cardinal numbers (used in aritmetical problems on objects) in both Ndau and Sena.

**The teacher of the experimental lessons**

The teacher's mother-tongue is Chuwabo. He had to learn the spoken numeration systems in Ndau and Sena.

During the 8-week programme, the teacher changed his teaching style: in the beginning he was convinced that he should always show how to do a certain computation and then have the pupils practice the same method. Later he learned to pose a problem and see how the children tried to solve the problem. Then in discussion with the pupils the best methods were chosen.

With the introduction of sums of two digit numbers exceeding ten, the teacher posed the problem *nine plus four* and left the pupils to try to find a solution. The teacher had the patience to let 11 children come in front of the class and show their solutions. In the end the teacher repeated the computation done by one of the pupils:

| | | |
|---|---|---|
| 1. Show *nine* using the left hand. First, show the open hand; put the five in the pocket and show *four* with the same hand:<br> | 2. Add *four*, using the other hand. Now we have two *fours*, and *five* in the pocket.<br> | 3. Two *fours* are transformed into *three* and *five*:<br><br>The *five* on the hands, plus the *five* in the pocket, make *ten*; plus *three*, *ten-and-three*, *thirteen*. |

The other methods presented by the pupils included: counting all, counting on from 9, and addition up-over-ten (9 + 4 = 9 + 1 + 3 = 10 + 3 = 13).

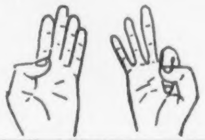
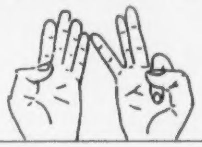
### Recomposition methods structured around five

During the lessons on addition and subtraction within the limit of ten it was found that the first advanced computing strategies are developed around five, similar to the recomposition methods around ten, but on a lower level.

*4 + 3 was computed in the three different ways:*

a) Show *four* on one hand. *Four* plus *one*, *five* (unfolding the thumb). *Five* plus *two*, *seven* (showing *two* on the other hand. (*recomposition up-over-five*)

b)

| | |
|---|---|
| **1.** Show *four* on one hand and *three* on the other:<br> | **2.** Separate the little finger from the *three*, touching the little finger of the *four* as a sign of transforming *four and three* into *five and two*, *seven*.<br> |

c) A second method of transforming 4+3 into 5+2 was done by taking away one finger of the *three*, folding the middle finger, and adding a finger to the *four* through the unfolding of the thumb.

In all three methods the pupils recognise that *five-and-two* is the same as *seven*, "by definition". Pronounced in Chuwabo, *tanu na pili* is the final answer.

*Subtraction*

Teachers and pupils found easily, with gestures, that the corresponding subtraction (7 – 3) may be solved, either by subtraction *down over five* (7 – 2 = 5; 5 – 1 = 4), or by *subtraction from five* (5 - 3 = 2; 2 + 2 = 4).

### Problems with subtraction

During the first lessons of gesture subtraction, a common mistake was that the pupils showed both terms of the difference, on different hands, and *added* the two numbers, instead of *subtracting* them. The problems were always presented orally, using the expression "take away" (and the quivalents in Ndau, Chuwabo and Sena).

A second difficulty with subtraction was with the gesture of *taking away some fingers*, which was clearly more difficult than *showing numbers*. Within a few days all children showed easily the numbers up to ten, with the correct number of fingers unfolded. But the inverse gesture – folding some fingers – was difficult for many children. The teacher suggested that the children could use one hand, in order to bend fingers on the other hand. It was even noted that apparently equal gestures could be different in the execution. E.g., in order to show 2 fingers, the pupil has to fold 3 fingers, starting with the thumb. But the pupil feels it like showing 2 fingers and not like bending 3 fingers. But when the same pupil has to show the subtraction 5 – 3 = 2, he has difficulties: he starts easily showing five unfolded fingers. But bending three fingers costs a special effort.

### On the reuse of fingers for numbers greater than 5 or 10

Fuson & Kwon indicate that one of the reasons why ten-structured methods are not common in the United States is that "there is not a culturally supported way to reuse fingers to show numbers larger than 10 when finding sums and differences to 18". With the gesture computation practiced in the experimental programme, children learned to represent on one hand numbers bigger than 5, and on two hands numbers greater than 10. E.g., they calculated 8 + 6 = 14 in the following way: Show *eight* on the left hand, raising the hand with the *five* fingers unfolded; raise the same hand again, with *three* fingers unfolded; in the end only *three* fingers are visible. Where are the five fingers of the eight? The children would say "the *five* is kept in the head". Then the children would show *six* on their right hand, i.e., show one full hand and say "I keep *five* in my head" and then show *one* more finger on the right hand, and say: "*five* (in the head) plus *one*, *six*". So the children end up showing 3 fingers on the left hand and 1 finger on the right hand and say: "*ten* in the head, plus *three* and *one*, gives *ten and four – fourteen*". During several lessons pupils used this method of reusing their fingers. Then a colleague who observed a lesson proposed to put the *five* in her pocket, instead of in her head. This method proved to be a success, probably because the action of put away a number was made by a concrete movement. This movement helped the children later to look at their pockets in order to remember if something was put in them. The gesture of putting 5 or 10 in a pocket was accompanied by the eyes (and the head). In a later phase, the action of putting something in a pocket was abbreviated and in the end only a slight movement of the head remained, indicating that the child remembered a number which is no longer visible.

### Language preferences

The pupils' attitudes towards the use of the different languages was varied:

a) Some pupils learned to verbalize the gesture computation in all four languages; others felt comfortable only with their mothertongue and with Portuguese; others had a clear preference for Portuguese. A few pupils continued to use their own language timidly – probably because of the general prohibition normal in schools (and maybe in some homes).

b) Sometimes, the teacher posed a problem in one language and a pupil answered in another. In general the teacher accepted it as something normal. Sometimes he asked the pupil to give the answer in the language which the teacher had used.

### General results

Pupils and teachers adopted easily the system of gestures in order to present the numbers from 1 to 20. The gestures seem to correspond to existing cultural habits.

Teachers and pupils had to experiment, in order to find easy ways to do addition and subtraction with gestures. Practically all ten-structured and all five-structured methods were found and used.

Verbalization of the different steps of gesture computation requires a conscious effort, because it is not necessary in order to find the results. Only in the long run the verbalization will be useful, when it takes the place of the gestures.

The programme was too short and started too late: if the representation by gestures and the use of the mothertongues had started in the beginning of the school-year, the results would have been very different. The teachers are eager to repeat the programme starting in the beginning of Grade 1.

Throughout the experimental programme, some pupils continued to use counting strategies, using their fingers or making strokes in their exercise books. However, the use of toes disappeared.

## REFERENCES

Baroody, Arthur J.: 1987, *Children's mathematical thinking – a developmental framework for preschool, primary and special education teachers*, Teachers College, Columbia University, New York – USA.

Coelho, Jorge J.T.R.: 1997, *Interview with Matude Lameque, February 15, 1997 in Espungabera*, Universidade Pedagógica, Maputo – Mozambique (unpublished report).

Draisma, Jan: 1998, 'On verbal addition and subtraction in Mozambican bantu languages', in Alwyn Olivier & Karen Newstead: *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, University of Stellenbosch, Stellenbosch – South Africa, Vol. 2, p. 272–279.

Draisma, Jan: 1999, 'Numeração falada e gestual como recursos naturais na aprendizagem inicial da Matemática' [Spoken and gesture numeration as resources for the early learning of Mathematics], in Comissão Organizadora do ProfMat 99, *Actas do ProfMat 99* [Proceedings of 15th National Meeting of the Association of Mathematics Teachers], Associação de Professores de Matemática, Lisbon – Portugal (p. 253–269).

Gerdes, Paulus (Ed.): 1993, *A numeração em Moçambique. Contribuição para uma reflexão sobre cultura, língua e educação matemática*, Instituto Superior Pedagógico, Maputo – Moçambique.

Gerdes, Paulus & Cherinda, Marcos: 1993, 'Sistemas africanos de numeração', in P. Gerdes (Ed.), *A numeração em Moçambique*, Instituto Superior Pedagógico, Maputo – Mozambique; p. 8-34.

GETEA – Inspecção Provincial de Educação: without date (but before 1975), *Didáctica das lições da classe pré-primária – Livro do Professor, Volume 2*, Secretaria Provincial de Educação, Lourenço Marques (now Maputo), Província de Moçambique – Portugal (!).

Hatano, Giyoo: 1982, 'Learning to Add and Subtract: A Japanese Perspective', in T.P. Carpenter, J.M. Moser & T.A. Romberg: *Addition and subtraction: A cognitive perspective*, Lawrence Erlbaum Associates, Hillsdale, New Jersey – USA (p. 211 – 223).

Fuson, Karen C. & Kwon, Youngshim: 1992, 'Korean children's single-digit addition and subtraction: numbers structured by ten', in *Journal for Research in Mathematics Education*, Vol. 23, N° 2, 148-165.

Kilborn, Wiggo: 1990, *Evaluation of textbooks in Moçambique. Mathematics grade 1 - 3*, University of Gothenburg, Gothenburg – Sweden.

Menninger, Karl, 1969: *Number words and number symbols*, M.I.T. Press, Cambridge (Massachusetts) – USA

Padberg, Friedhelm: 1986, *Didaktik der Arithmetik*, Bibliographisches Institut, Mannheim/ Wien/ Zürich – Germany/Austria/Switzerland.

INDE – Instituto Nacional do Desenvolvimento da Educação: 1989, *Programa de Matemática. Ensino primário – 1ª à 7ª classes*, INDE, Maputo – Mozambique.

Treffers, A.; De Moor, E.: 1990, *Proeve van een nationaal programma voor het reke-wiskundeonderwijs op de basisschool. Deel 2 - Basisvaardigheden en cijferen*. [Proposal of a national syllabus for the teaching of arithmetic-mathematics in the basic school], Zwijsen, Tilburg – The Netherlands.

# STUDENTS' STATISTICAL REASONING DURING A DATA MODELLING PROGRAM

Lyn D. English, Kathy L. Charles, Queensland University of Technology
Donald H. Cudmore, Oxford University

*This paper reports on the first year of a 3-year international project addressing ninth- and tenth-grade students' statistical reasoning processes as they participate in a range of data modelling experiences. These experiences engage students in constructing statistical problems and cases, as well as in reasoning critically and philosophically. Among the first-year findings is the need to develop further students' (a) knowledge of statistical measures and data representations to enable them to construct problems and cases that are not statistically naïve; and (b) abilities to reason more critically about the statistical issues and arguments they are advancing.*

The importance of fostering students' statistical reasoning processes has been highlighted by both the research literature and recent curriculum documents (e.g., Australian Association of Mathematics Teachers, 1996; Cobb, 1999; Greer, in press; Gal & Garfiel, 1997; Hancock, Kaput, & Goldsmith, 1992; National Council of Teachers of Mathematics, 1998; Shaughnessy, Garfield, & Greer (1996). In addition to preparing students to cope with the increasing use of data and data-based arguments in society, statistical reasoning activities provide meaningful contexts for the application of basic mathematical ideas, and can help students develop an appreciation of mathematics as a way of interpreting their world (Hancock et al., 1992).

Accompanying these calls for students' statistical development is a focus on student-driven classroom projects (Hancock et al., 1992), where students are given opportunities to engage fully in the practices and processes of meaningful statistics (Derry, Levin, Osana, & Jones, 1998; Lehrer & Romberg, 1996; Moore, 1998). A particularly important type of classroom project that has been emphasized in the literature but has received limited research in the elementary and middle schools is data modelling. Lehrer and Romberg (1996) referred to data modelling as the "construction and use of data" where "the very idea of data entails a separation between the world and a representation of that world" (p. 70). Hancock et al. (1992) adopted a broader perspective, viewing data modelling as "a complete process of inquiry"(p. 338). Likewise, we view data modelling as encompassing students' active engagement in collecting, interpreting, and refining data; in representing data and accompanying explanations in an appropriate, convincing form; and in posing and critiquing statistical problems and cases. Little is known, however, about students' reasoning processes when engaging in data modelling activities (Lehrer & Romberg, 1996). In particular, there are few studies addressing students' construction of their own problems for others to solve and critique based on data generated from classroom projects (Hancock et al., 1992; Jacobs, & Lajoie, 1994). Yet problem construction is a natural component of authentic data-based projects, especially when students and teachers work within a collaborative learning environment that values and encourages a diverse range of reasoning processes (Lehrer & Romberg, 1996; Scardamalia & Bereiter, 1992).

In developing the present study, we identified a number of reasoning processes considered important in developing students' statistical competence (Gal & Garfield, 1997; Derry et al., 1998; Scheaffer, Watkins, & Landwehr, 1998; Watson, 1997). We were particularly interested in observing developments in these processes as classes of ninth-graders participated in our modelling program We addressed the following:

**Understanding and reasoning about the processes of statistical inquiry**: This includes an understanding and appreciation of the phases of a statistical inquiry (e.g., formulating questions, planning a study, collecting, organizing, and analyzing data).

**Reasoning about data**: This entails knowing that raw data need to be reduced to facilitate interpretation, and being able to describe, compare, and explain sets of data.

**Reasoning about statistical measures**: Such reasoning involves an understanding of what measures of central tendency tell about data sets, knowing which measure (s) is best to use under different conditions, and knowing how to utilize measures of centre and spread for comparing data sets.

**Reasoning about representations of data**: This includes understanding how data are summarized and presented (Lajoie et al., 1998), together with determining the appropriateness of the tables, charts and/or graphs constructed to represent data. To reason effectively about data representations, students must be able to reason about data and understand how to read, construct, and interpret a graph/table/chart; they must also know how to modify these representations to better represent a given data set.

**Reasoning and communicating in a critical and philosophical manner**: This includes: (a) being aware of possible biases or limitations in the generalizations that can be drawn from data; (b) being able to use statistical terminology appropriately, convey results in a convincing manner, and develop reasonable arguments based on data or observations; (c) being able to challenge the validity of the data interpretations and representations of others; and (d) being able to critically evaluate statistical information, to skilfully defend or reject particular statistical arguments, to effectively critique statistical problems posed by others, and to think flexibly and creatively in extending and modifying statistical problems.

## Method

**Participants**. Two classes of ninth-grade students (aged from 13 to 15 years) and their teachers from a lower-middle class Australian metropolitan school participated in the present study. We focus here on the second class (N=26), who completed our program in the fourth term of the school year. Embedded within the large class case study were six cases (Yin, 1989), whose members were observed in fine detail throughout the program. The cases were selected on the basis of two tests of statistical knowledge and reasoning. One test focused on students' knowledge of basic statistical ideas and procedures (X= 10.9 out of 25; SD = 3.03), while the second on their statistical reasoning processes (X= 2.2 out of 10; SD = 1.7). We consider here Camina and Christie who scored below the class mean on both tests; Damien who scored the highest class mean on the second test, but scored just above the mean on the first test; and Julian and Vicki who scored above the class mean on both tests.

**Program**. Our program followed Lajoie et al.'s (1998) approach of employing multiple contexts within which students could develop their statistical understandings and reasoning processes. We worked collaboratively with the classroom teacher in implementing the program, ensuring that a community of learners was fostered (Cobb & Bowers, 1999). We encouraged students to work collaboratively and to express their ideas and sentiments in an open and constructive manner. The students were provided with the appropriate technological supports and a website where (a) the researchers and teachers shared thoughts on the progress of the program, and (b) students posted their responses to the activities, including their statistical problem constructions. The program comprised 16 sessions (approx. 70 mins. each), with three phases as follows.

**Phase 1: Reasoning about data, and reasoning critically and philosophically.** Six lessons were devoted to this component in preparation for the remaining phases. The students were introduced to data-based inquiry by exploring existing data from a previous survey (located in our website) that had been constructed and completed by grade nine classes in other schools (Cudmore & English, 1998). The students reasoned about data types, issues of sampling, approaches to organizing and reducing data, measures of central tendency, and the use basic statistical software. As the students worked with the data, they identified patterns and trends, made conjectures, drew inferences, and engaged in philosophical discussions on contentious issues (e.g., on why Australian students favored the monarchy more than the UK students). At the end of this phase, the students applied their statistical understandings to a critical analysis of a statistically flawed newspaper article on computer use in schools (Watson, 1997).

**Phase 2: Understanding and reasoning about the processes of statistical investigations; reasoning about data, data representations, and statistical measures.** This phase comprised three sessions devoted to students' construction of their data gathering instrument, namely, a student-constructed survey. Working in small groups, the students brainstormed survey questions for consideration for inclusion in their new survey. The students were encouraged to consider carefully the nature and types of questions they were constructing (e.g., whether the questions would be of interest to their peers, and whether the questions would generate different types of data). A class discussion followed in order to select questions for the final survey, which was placed on our project website for each student to complete. Discussion followed on how the data might be collated and analyzed. Where necessary, the students were assisted in collating the data into an Excel spreadsheet.

To guide the students in analyzing the data, and to help prepare them for the problem-posing activity of the next phase, we engaged them in an activity titled, "We noticed, we wondered." Working in groups, the students addressed the two statements: "In exploring the data, we noticed that . . ." and "We then wondered . ." This activity required the students to make informed statements about the data, to draw inferences, and to make conjectures.

**Phase 3: Applying all reasoning processes**. Seven sessions were devoted to this phase, which comprised two types of data modelling activities. The first type required the students to select a question from the survey and construct a comprehensive problem that was solvable using the survey data. In preparation for this activity, the students discussed a number of issues regarding problem construction, including (a) How would you define a mathematical problem? (b) What features does a problem require for it to be considered a mathematical problem? an appealing mathematics problem? a challenging mathematics problem? (c) What types of questions might you ask the solver? (d) Would you ask the solver to include a table or graph in their working of your problem? (e) Would your problem have more than one answer? After generating their problems, the students critically analyzed one another's constructions, made modifications to their problems based on this feedback, and then posted their final problems on the website for others to try.

The second data modelling activity required the students to construct a statistical case for presentation to their school council or local community that argued for a particular stance on a selected issue. The issue was to be drawn from the survey questions, with the case being supported by the students' analysis of the data (e.g., after analysing selected data, the students might present a case that argued for improvements to the school uniform). The students were encouraged to support their case with appropriate visual representation, to make some recommendations for further action, and to highlight any related issue/s for further research.

We used a range of field-based methods to obtain evidence of the students' developments in statistical reasoning, including field notes, videotapes of the students' discussions, explanations, and comments, their responses on the website, and informal student and teacher interviews. We also recorded on our website our observations of the progress of each lesson immediately following its completion.

## A Selection of Findings

### Constructing and reasoning about data

Many of the students' initial reasoning with measures of central tendency (mean, mode, and median) was largely proceduralized, with the students more concerned about selecting the correct spreadsheet formula than whether they were using the appropriate measure for the data type (cf. findings of Cudmore, 1999; Watson & Moritz, 2000). For example, one group of students recorded a mean of 6.75 for categorical data and blindly accepted this. Although some students disagreed over which statistical measure to use, they could not offer reasons for why one was more appropriate than another.

In phase 2, the students were cognizant of posing a variety of question types for their survey construction, but frequently overlooked the data encoding difficulties presented by open-ended questions (cf. Lehrer & Romberg, 1996). For example, Vicki motivated her group by suggesting that they generate "*a question relative to today*," and asked them if they wanted to "*start with numerical questions or categorical questions*." Damien suggested some open-ended questions, such as "Name your favorite band." Others enthusiastically endorsed such questions, but Vicki alerted the group to the potential problems: "*It's going to be like a huge . . . I mean, think of it as like, they've got to write down—how many groups are we going to get across the page? Let's write, 'Out of these, who is your favorite group?' Do you want to just put down five groups?*" Although the students took heed of Vicki's advice, their focus on constructing appealing questions soon overtook any concerns about data encoding. Even Vicki reverted to questions such as, "How far do you travel to school?" and "How often do you use public transport?". Finally, Damien made his group aware of the need for refining their questions and listing suitable sub-categories.

The students did, however, demonstrate a critical awareness of the semantics of their survey questions. For example, for the question, "Do you think smoking should be banned?" Damien asked, "*Where. . . the whole of Australia, the whole world, or just school?*" to which another student replied, "*So what is the question, like, is it should be banned . . . the word is should, so do we think smoking should be banned?*" A third student commented that smoking is already banned in their school, so "*if we're saying it 'should be banned,' that's saying it's not banned already.*"

By the end of phase 2, the students were able to reason more critically about the data they were handling. For example, when analyzing the findings from one of the survey questions, namely, "How much pocket money do you get a week?" they found the mean was $9 per week and the mode, $5. When asked to explain what these meant, Damien commented, "*Well, it's nice to know that you're not that low as everyone else.*" In responding to a question about the measure they might use if they wanted to persuade their parents to increase their pocket money, Vicki replied, "*If I was going to show my parents the average amount of pocket money so they'd give me more pocket money, I'd show them 9.*" *Damien added, "Yeah, I'd show them the mean, not the mode cause it's lower, and so is the median. . . . that's because someone put $40* (in completing the survey). . . *maybe they earn money; maybe they have a job.*"

By the end of phase 3, the majority of students were able to apply their understanding of statistical measures and of problem design to their construction of statistical problems and cases. They were better able to construct statistical problems than cases, which is perhaps not surprising given that students tend to become enculturated in the world of mathematical word problems (Gerofsky, 1996). The students were able to create appealing and diverse contexts in their problem constructions, as can be seen in

Melissa's problem: Two aliens were in their spaceships looking down on Earth. "We have to let the inhabitants know that we're here," said the first alien. "The only way we have to communicate with them is through the television," replied the second alien. "We have to choose a time that most of the inhabitants are watching the television, so we can interrupt their program. This means we will have to find out the humans' favorite television show." To help the aliens communicate, find out the favorite TV show from the survey results. Show the results in a frequency distribution table and then turn it into a histogram. What is your favorite TV show? However, the majority of students created statistically naïve, albeit solvable, problems. That is, 70% of the students focused solely on frequencies and finding the mode in their problems; few addressed other statistical measures. The students also rarely referred to the formal terms, mean, mode, and median. While the majority of students (91%) asked the solver to make a decision or discovery about the data, only 3 students included an additional question to extend their peers' reasoning even though we had discussed this aspect in class (e.g., "Do you think the school is doing the right thing by upgrading their transport facilities or should the money be spent on something else?") Nevertheless, 50% of the students created problems that involved more than one variable, with 6 students addressing relationships between variables. Interestingly, when the students critiqued one another's problems, the most common criticism was that the problem was too easy.

With respect to the statistical cases, 92% of the class was able to present a case using the survey data, with 60% reporting appropriately one or more explicit measures of central tendency and using these to justify their arguments. However, only 50% of the students referred to the sample size in explaining their case and only six students commented that additional data collection would further support their case.

### Students' reasoning about data representations

During the first phase, the students generally displayed limited knowledge of data representations. Of the case studies, Julian and Damien displayed the greatest understanding, with Julian stating that a pie graph "*shows portions*," and a line graph "*is for time; it shows increase or decrease in data.*" Camina and Christie viewed graphical forms in terms of their surface features ("*A pie graph is a circle, a line graph has lines, and a bar graph has bars*"), although Camina did add to Julian's notion of a pie graph: "*Yes, it shows percentages.*" Most of the students did not consider the importance of selecting an appropriate representation for given data. Damien was the exception here. When his group was trying to construct a graph to illustrate some survey data, he advised his peers: "*We want the values, we don't want the formulas . . . we want a graph to go in here . . . we have choices . . ., just let me think what would be best for this data . . .*" In response to another student's suggestion to "*Just do every chart there . . . you can't go wrong, can you? You'll find one sooner or later,*" Damien looked askance, stating, "*We'll choose that one . . . now, what you have to do is make sense of what those figures mean and what the graph is telling you. So have a look at where the data came from and what will that mean? How is it helping you to understand the data?*"

In using data representations to support their statistical cases (phase 3), just less than half the students constructed an appropriate representation and only two students used more than one form. Likewise, in constructing their statistical problems, only 39% of students asked the solver to provide some form of visual representation. Students' reluctance to work with data representations could be due to their perception that it is "*too time consuming*," as Julian had argued previously. Damien again was the exception, claiming that graphs indicate that one has "*thought through the question*" and "*at least understood it a bit.*" Yet Damien was concerned that if a spreadsheet were unavailable to his peers in solving his problem, they would be unable to construct a graph. His fears failed to materialize; those students who chose to draw representations by hand in solving a peer's problem constructed appropriate and complete representations (in contrast to others who failed to do so when using a spreadsheet to accomplish the task).

### Reasoning and communicating in a critical and philosophical manner

Perhaps not surprisingly, the students were more willing to engage in critical discussion about controversial and relevant issues that arose from their survey findings than they were about statistical issues such as discussing their choice of statistical representation. Of all the students, Vicki stood out as one of the most enthusiastic and insightful discussants. She expressed her views readily, and speculated on why certain trends appeared in some of the data. For example, in their discussions about a previous survey finding (a question asking students to rate their concern about the environment) Damien commented, "*We could say that most people don't really have much to do with preserving their environment.*," to which Vicki replied, "*Yeah, true. It's always an issue but it's not always a concern . . . the ones that are very concerned, well it may be that they're watching the news or something and seen something that disturbed them. . . We could just sum it up by saying there's no major difference between the schools. Like you can't define what it is ... the only thing we can do is like speculate why they chose the responses they picked.*" Vicki also strongly defended her beliefs. When students were sharing their statistical cases, she was very vocal in disagreeing with a male student's case on why boys participate in more school sports than girls. Despite her teacher (a female) supporting the male students' case, Vicki claimed his arguments were "*not right; they're stereotyped.*"

## Concluding Comments

As the students became further immersed in the program activities, they were generally able to reason in a more meaningful way about the data they were handling. However, their knowledge of statistical measures and data representations still required further development to enable them to pose problems and cases that were not statistically naïve. Importantly, the students also needed to reason more critically about the statistical issues and arguments they were advancing. Students' abilities to engage in philosophical discussion about topical, controversial issues need to be nurtured in the statistical domain. Students such as Damien, whose statistical reasoning skills could be easily overlooked in regular class tests, alert us to the importance of fostering a range of reasoning processes in statistics education.

## References

Australian Association of Mathematics Teachers (1996). *Mathematical knowledge and understanding for effective participation in Australian society.* A National Statement on Mathematics for Australian Schools. AAMT.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning, 1*, 5-43.

Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspectives in theory and practice. *Educational Researcher, 28*(2), 4-15.

Cudmore, Donald H., (1999), *A Study of Secondary Mathematics Students Engaged in Problem Posing and Critiquing Activities*, Unpublished doctoral dissertation, University of Oxford

Cudmore, D. H., & English, L. D. (1998, April). *Using Intranets to foster statistical problem posing and problem critiquing in secondary mathematics classrooms.* Paper presented at the annual conference of the American Educational Research Association, San Diego.

Derry, S. J., Levin, J. R., Osana, H. P., & M. S. Jones (1998). Developing middle school students' statistical reasoning abilities through simulation gaming. In In S. P. Lajoie (Ed.), *.Reflections on statistics: Learning, teaching, and assessment in grades K-12* (pp. 175-195). Mahwah, NJ: LEA.

Gal, I., & Garfield, J. B. (1997). Curricular goals and assessment challenges in statistics education. In I. Gal & J. B. Garfield, (Eds.). *The assessment challenge in statistics education* (pp. 1-13). Amsterdam: IOS Press.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics, 16*(2), 36-45).

Greer, B. (2000). Statistical thinking and learning: Introduction. *Mathematical Thinking and Learning, 2*(1&2). Special issue on Statistical Thinking and Learning.

Hancock, C., Kaput, J. J., & Goldsmith, L. T. (1992). Authentic inquiry with data: Critical barriers to classroom implementation. *Educational Psychologist, 27*(3), 337-364.

Jacobs, V., & Lajoie, S. (April, 1994*). Statistics in the middle school: An exploration of students' informal knowledge*. Paper presented at the annual meeting of AERA, New Orleans.

Lajoie, S. J., Lavigne, N. C., Munsie, S. D., & Wilkie, T. V. (1998). Monitoring student progress in statistics. In S. P. Lajoie (Ed.), *.Reflections on statistics: Learning, teaching, and assessment in grades K-12* (pp. 199-232). Mahwah, NJ: Lawrence Erlbaum Associates.

Lehrer, R., & Romberg, T. (1996). Exploring Children's Data Modeling. *Cognition and Instruction, 14*(1), 69-108.

Moore, D. S. (1998). Statistics among the liberal arts. *Journal of the American Statistical Association, 93*, 1253-1259.

National Council of Teachers of Mathematics (1998). *Principles and standards for school mathematics : Discussion draft.* Reston, VA:National Council of Teachers of Mathematics.

Scardamalia, M., & Bereiter, C. (1992). Text-based and knowledge-based questioning by children. *Cognition and Instruction, 9*(3), 177-199.

Scheaffer, R. L., Watkins, A. E., & Landwehr, J. M. (1998). What every high-school graduate should know about statistics. In S. J. Lajoie (Ed.), *Reflections on statistics: Learning, teaching, and assessment in grades K-12* (pp. 3-26). Mahwah, NJ: Lawrence Erlbaum Associates.

Shaughnessy, J. M., Garfield, J., & Greer, B. (1996). Data handling. In A. Bishop, K. Clements, C. the media. In I. Gal & J. B. Garfield, (Eds.). *The assessment challenge in statistics education* (pp. 107-121). Amsterdam: IOS Press.

Watson, J., & Moritz, J. (2000). The longitudinal development of understanding of average. *Mathematical Thinking and Learning, 2*, (1&2). Special issue on Statistical Thinking and Learning.

Yin, R. (1989). Case study research: Design and methods. London. Sage Publications.

BEST COPY AVAILABLE

# THE 'MATHEMATICS AS A GENDERED DOMAIN' SCALE

Helen J. Forgasz
Deakin University, Australia

Gilah C. Leder
La Trobe University, Australia

***Abstract***

*The Fennema-Sherman Mathematics Attitude Scales [MAS], first published in 1976, are widely used to determine students' attitudes towards mathematics. The construct underpinning one of the scales, Mathematics as a Male Domain [MD], is considered critical in explaining subtle yet persistent gender differences in mathematics learning outcomes. In an earlier (1996) PME paper we argued that with the changes that have occurred in recent years in society's beliefs and expectations of men and women, the underlying assumptions upon which the MD scale items were developed can no longer be regarded as valid. In this paper we report on the development of a new instrument to measure gender stereotyping of mathematics, and the results we obtained when this instrument was administered to a sample of approximately 850 students in grades 7 to 10.*

## Introduction

Mathematics education researchers interested in gender issues have used the Fennema-Sherman [F-S] *Mathematics Attitudes Scales* [MAS] (Fennema & Sherman, 1976) extensively to measure attitudes towards mathematics. The MAS consist of nine scales: confidence; effectance motivation, mathematics anxiety, usefulness of mathematics; attitude towards success; mathematics as a male domain [MD]; and father, mother and teacher scales. These constructs are "hypothesized to be related to the learning of mathematics by all students and/or cognitive performance of females" (Fennema & Sherman, 1976, p.1). In Fennema and Sherman's (1977) first reported study about the MAS, (a paper which is among the most cited articles in mainstream journals of educational psychology) *Mathematics as a Male Domain* scale findings revealed that:

> Male responses differed significantly from female responses. While boys did not stereotype mathematics strongly as a male domain on this scale, they always stereotyped it more strongly than did females. (p.68)

An extensive meta-analysis of mathematics education research studies incorporating affective variables and gender revealed that "mathematics as a male domain" had the greatest gender difference (i.e., largest effect size) (Hyde, Fennema, Ryan, Hopp & Frost, 1990). Gender differences in perceptions of mathematics as a male domain were shown to have declined during the 1980s, but were still evident in current work.

The MAS were published in 1976. The assumptions underpinning the development of the MD were noted by Fennema and Sherman (1976):

> The less a person stereotyped mathematics, the higher the score. This is done to fulfill the purpose of the scale development as it was assumed

> that the less a female stereotyped mathematics as a male domain, the more apt she would be to study and learn mathematics. (p. 7)

The corollary of this assumption is, presumably, that low-scoring females believe mathematics to be a male domain and might thus be less likely to study and learn mathematics. When the scale was developed, there was no apparent allowance for beliefs that mathematics might be considered a *female* domain. This approach was consistent with prevailing Western societal views of the 1970s, but, as we argued in earlier papers (Forgasz, Leder, & Gardner, 1996; 1999), such an assumption can no longer be supported today.

**A new instrument**

Two versions of a new instrument aimed at measuring beliefs about the stereotyping of mathematics as a gendered domain – the extent to which it is believed that mathematics is more suited to males, to females, or is regarded as gender-neutral – have been developed in an attempt to address the criticisms of the original Fennema-Sherman *Mathematics as a male domain* subscale.

In developing the items for both instruments, we drew on previous research findings about gender issues in mathematics learning – perceptions of ability, gender-appropriateness of careers, general attitude towards mathematics (e.g., enjoyment, interest), environment (e.g., teachers, classrooms, parents), peer effects, effort and persistence, and perceptions about mathematical tasks (e.g., difficulty) (e.g., Burton, 1990; Fennema & Leder, 1993; Leder, Forgasz, & Solar, 1996). Feedback was obtained from 10 volunteer mathematics educators and some two dozen volunteer grade 7 to 10 students. Various items were omitted or further modified as a result.

In this paper, we describe in some detail for the first time information for one version of the new instrument, the *Mathematics as a gendered domain* scale. (see Forgasz, Leder & Kaur, 1999, for details of the other version). For the *Mathematics as a gendered domain scale*, a traditional Likert-type scoring format was used – students were asked to indicate the extent to which they agreed (or disagreed) with each statement presented. A five-point scoring system was used – strongly disagree (SD) to strongly agree (SA). A score of 1 was assigned to the SD response and a score of 5 to SA. The instrument consisted of **48** items. It contained three subscales: *Mathematics as a male domain* [MD], *Mathematics as a female domain* [FD], and *Mathematics as a neutral domain*[ND]. There were 16 items for each subscale which were presented in a random order on the instrument. The question 'How good are you at mathematics [HGM]?' was included on the instrument; there were five response categories: excellent (scored at 5) – weak (scored at 1). (To conform with space constraints, data from this item are not presented in this paper.). A space was also provided for students to comment on any aspect of the instrument and its items.

In the first trial, approximately 200 grade 7-10 students from Victorian (Australia) schools completed the instrument. Statistical tests were conducted to determine the effectiveness of the different items. Psychometrically unsatisfactory items were deleted and a few others added to produce the second version of the instrument. This version was administered in May-June of 1999 to 846 grade 7-10 students from eight co-educational schools situated in the metropolitan and country regions of Victoria. Of these, 412 were females, 408 were males, with sex not specified on the remaining 26 questionnaires. Sample items from the final version of the three subscales of the *Mathematics as a gendered domain scale* are shown on Table 1.

Table 1. Selected items from the *Mathematics as a gendered domain* scale

| SUBSCALE | ITEM |
|---|---|
| Male domain – ability | Boys understand mathematics better than girls do |
| Female domain – career | Girls are more suited than boys to a career in a mathematically-related area |
| Neutral domain – general attitude | Students who say mathematics is their favourite subject are equally likely to be girls or boys |
| Male domain – environment | Boys are encouraged more than girls to do well in mathematics |
| Female domain – peers | Boys are distracted from their work in mathematics classes more than are girls |
| Neutral domain – effort | Girls and boys are just as likely to be lazy in mathematics classes |
| Male domain – task | Boys, more than girls, like challenging mathematics problems |

### Psychometric properties of the new instrument

A *confirmatory factor analysis* was conducted to check that the scale did consist of three distinct (orthogonal) subscales. The results of the Varimax rotation are shown in Figure 1. Three clear factors comprising the items on each of the three subscales were identified. It should be noted that subscale identity is designated by the first two letters of the item name (e.g., MDAbil1 – an 'ability' item on the *Mathematics as a male domain* subscale).

A reliability analysis was conducted on the items comprising each subscale. For each subscale, item-total correlations (no value <.3) confirmed the internal consistency of the items. Cronbach Alpha values for the three subscales were as follows:

- MD: ∀ = .902
- FD: ∀ = .897
- ND: ∀ = .836

### Descriptive statistics

The mean score for each of the three subscales was calculated (range 16-80). These scores were divided by 16 (number of items per subscale) giving scores between 1 and 5 (values consistent with the Likert-scale scoring to assist in the interpretation of the findings). The subscale means and standard deviations are shown on Table 2. The data reveal that, in general, students did not gender stereotype mathematics. Mean scores were > 3 for perceptions of mathematics as a *neutral domain* and < 3 for perceptions of mathematics as both a *male domain* and a *female domain*. Of interest here is the finding that students believed slightly more strongly that mathematics was a *female domain* than a *male domain*.

Table 2. Subscale means and standard deviations

| Scale | N | Mean | SD |
|---|---|---|---|
| **MD** | 736 | 2.33 | .670 |
| **FD** | 750 | 2.70 | .697 |
| **ND** | 738 | 3.84 | .556 |

Bivariate Pearson product-moment correlations were found for each combination of subscales. The results are shown on Table 3.

Table 3. Pearson correlations

| | FD | ND |
|---|---|---|
| **MD** | .468** | -.367** |
| **FD** | | -.367** |

* p<.05 ** p<.01

The significant negative correlation between the ND scale and the MD scales is consistent with beliefs that mathematics is either a neutral domain or a male domain; the significant negative correlation between the ND and the FD is similarly explained. The significant positive correlation between the MD and the FD scale is more problematic. A scatter diagram for scores on both scales (Figure 2) reveals that the combination of low scores (<3) on both scales – meaning general disagreement that mathematics is stereotyped as either a male or female domain – contributes to this correlation value.

### Gender differences

Independent t-tests were conducted to examine the data for gender differences. The results are shown on Table 4.

Table 4. Independent groups t-tests

| Scale | | N | Mean | |
|---|---|---|---|---|
| **MD** | F | 371 | 2.12 | *** |
| | M | 357 | 2.54 | |
| **FD** | F | 374 | 2.69 | |
| | M | 367 | 2.71 | |
| **ND** | F | 369 | 3.92 | *** |
| | M | 360 | 3.76 | |

*** p<.001

On two scales, MD and ND, statistically significant gender differences were noted. On average, males believed more strongly than females that mathematics was a *male* *domain* and females believed more strongly than males that mathematics was a *neutral domain*.

There were no statistically significant differences in males' and females' perceptions of mathematics as a *female domain*.

**Student comments**

Several students wrote comments in the space provided on the questionnaire. On the whole the comments were thoughtful and implied that the students had taken the task seriously, although some claimed the questionnaire was "sexist", "stupid", or a "waste of time". Of those who commented about the stereotyping of mathematics, more students reflected beliefs that mathematics was a neutral domain than either a male or a female domain. Some students were aware that it was politically incorrect to stereotype. Representative examples of student comments are reproduced below; sex and grade level are shown in brackets:

- Each boy is different and each girl is different. It is impossible to say girls like maths more than boys because no two people are the same. (M, Gr.7)
- Boys are as careful when it comes to maths as girls are. (F, Gr.9)
- Boys and girls are generally equal. It all depends on the individual. (M, Gr.10)
- My brother always says that he is smarter than me because I'm a dumb girl. (F, Gr.9)
- Although some girls' attitudes are different, girls have just as good a chance to excel at maths as boys. (M, Gr.8)
- I think that girls are better than boys at maths. (M, Gr.8)
- Boys at my school don't seem to care about mathematics or any other subjects as much as girls do. I think this is a result of peer group pressure. (F, Gr.9)
- These days I believe girls are trying harder to achieve top marks and they are getting them. Boys just don't care, they muck around in class and distract others. (F, Gr.9)
- Some of the questions are the same as other questions. Some questions are sexist. (M, Gr.9)
- You can't take a stereotyped approach to these type of questions these days. (F, Gr.10)

**Concluding comments**

In this paper we presented the rationale and methods used to develop a scale useful for investigating to what extent mathematics continues to be considered as a gendered domain. In a clear departure from the approach adopted by Fennema and Sherman (1976), the scale contained items suggesting that mathematics could be considered as a female domain, as well as items signifying mathematics to be a male or neutral domain. There were three separate subscales to measure the extent of these beliefs.

When the 48 item scale was administered to some 850 students, of ages comparable to the sample used for validating the original scale devised by Fennema and Sherman

(1976), three distinct subscales emerged – as anticipated. Each was found to have high internal consistency: the Cronbach Alpha values ranged from .836 to .902.

Collectively, the students agreed more strongly that mathematics represents a neutral domain than either a male or female domain. As in previous research, females rejected the notion of mathematics as a male domain more strongly than did males.

Inspection of student comments indicated that some students, males as well as females, now describe females as better at mathematics than males, consider that females are more prepared to work hard at mathematics, and that males are distracted and less concerned about succeeding. Thus it appears from our data that at least some students now regard mathematics as a female domain (though many more regard it as a neutral domain). Continued use of the original instrument devised by Fennema and Sherman (1976) would not have yielded this information.

Our study was conducted in one state in Australia. It is important to confirm its findings with data obtained from international and culturally diverse samples.

## References

Burton, L. (Ed.) (1990). *Gender and mathematics. An international perspective.* London: Cassell.

Fennema, E., & Leder, G. C. (Eds.) (1993). *Mathematics and gender.* St. Lucia, Queensland: Queensland University Press.

Fennema, E., & Sherman, J. (1976). Fennema-Sherman mathematics attitude scales. *JSAS: Catalog of selected documents in psychology*, *6*(1), 31 (Ms. No. 1225).

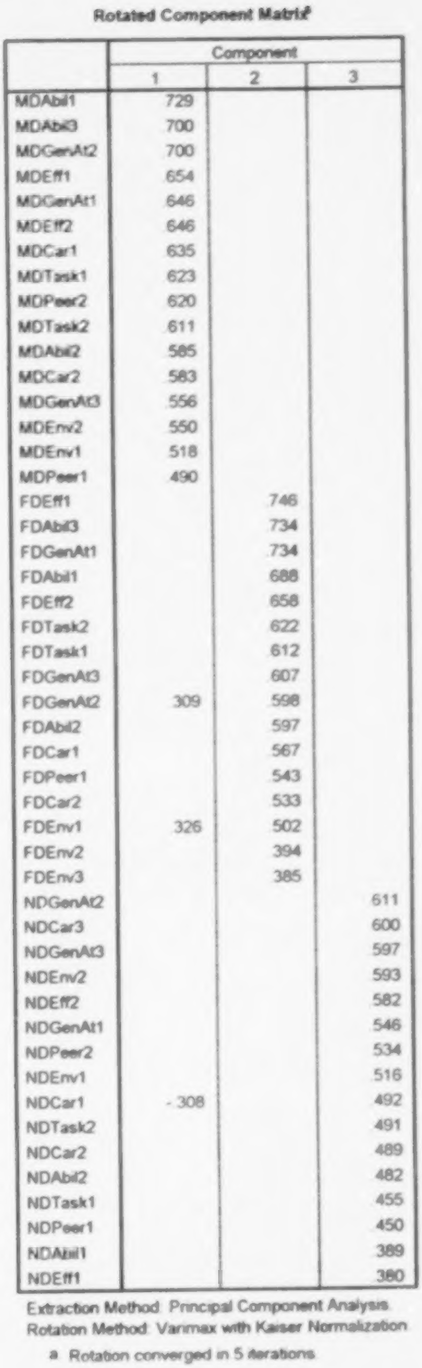
Fennema, E. & Sherman, J. (1977). Sex-related differences in mathematics achievement, spatial visualization and affective factors. *American Educational Research Journal*, *14*(1), 51-71.

Forgasz, H. J., Leder, G. C., & Gardner, P. L. (1996). The Fennema-Sherman 'Mathematics as a male domain' scale: A re-examination. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, Vol.2*, (pp.2-361 - 2-368). Valencia, Spain: International Group for the Psychology of Mathematics Education.

Forgasz, H. J., Leder, G. C., & Gardner, P. L. (1999). The Fennema-Sherman 'Mathematics as a male domain' scale re-examined. *Journal for Research in Mathematics Education*, *30*(3), 342-348.

Hyde, J. S., Fennema, E., Ryan, M., Frost, L. A., & Hopp, C. (1990). Gender comparisons of mathematics attitudes and affect: A meta-analysis. *Psychology of Women Quarterly*, *14*, 299-324.

Leder, G. C., Forgasz, H. J., & Solar, C. (1996). Research and intervention programs in mathematics education: A gendered issue. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education, Part 2* (pp.945-985). Dordrecht, Netherlands: Kluwer.



**Rotated Component Matrix**[a]

| | Component | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| MDAbil1 | .729 | | |
| MDAbil3 | .700 | | |
| MDGenAt2 | .700 | | |
| MDEff1 | .654 | | |
| MDGenAt1 | .646 | | |
| MDEff2 | .646 | | |
| MDCar1 | .635 | | |
| MDTask1 | .623 | | |
| MDPeer2 | .620 | | |
| MDTask2 | .611 | | |
| MDAbil2 | .585 | | |
| MDCar2 | .583 | | |
| MDGenAt3 | .556 | | |
| MDEnv2 | .550 | | |
| MDEnv1 | .518 | | |
| MDPeer1 | .490 | | |
| FDEff1 | | .746 | |
| FDAbil3 | | .734 | |
| FDGenAt1 | | .734 | |
| FDAbil1 | | .688 | |
| FDEff2 | | .658 | |
| FDTask2 | | .622 | |
| FDTask1 | | .612 | |
| FDGenAt3 | | .607 | |
| FDGenAt2 | .309 | .598 | |
| FDAbil2 | | .597 | |
| FDCar1 | | .567 | |
| FDPeer1 | | .543 | |
| FDCar2 | | .533 | |
| FDEnv1 | .326 | .502 | |
| FDEnv2 | | .394 | |
| FDEnv3 | | .385 | |
| NDGenAt2 | | | .611 |
| NDCar3 | | | .600 |
| NDGenAt3 | | | .597 |
| NDEnv2 | | | .593 |
| NDEff2 | | | .582 |
| NDGenAt1 | | | .546 |
| NDPeer2 | | | .534 |
| NDEnv1 | | | .516 |
| NDCar1 | -.308 | | .492 |
| NDTask2 | | | .491 |
| NDCar2 | | | .489 |
| NDAbil2 | | | .482 |
| NDTask1 | | | .455 |
| NDPeer1 | | | .450 |
| NDAbil1 | | | .389 |
| NDEff1 | | | .380 |

Extraction Method: Principal Component Analysis.
Rotation Method: Varimax with Kaiser Normalization.

a. Rotation converged in 5 iterations.

Figure 1. Varimax results for confirmatory factor analysis

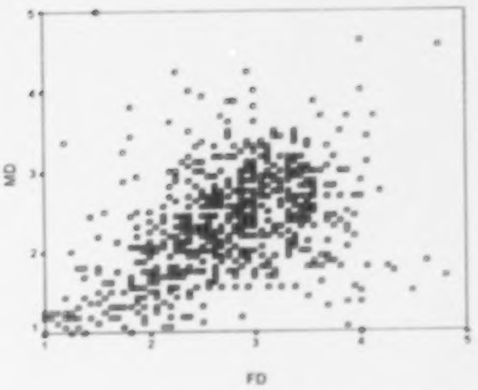


Figure 2. Scattergram for MD & FD score

BEST COPY AVAILABLE

# INVESTIGATING FUNCTION FROM A SOCIAL REPRESENTATION PERSPECTIVE

**Janete Bolite Frant, Monica Rabello de Castro, Flavio Lima**

**Affiliation – Universidade Santa Úrsula – Instituto de Educação Matemática**

## Abstract

We present an investigation for meaning production of function and a model based on argumentation and social representation theories for analyzing it. This investigation took place in a middle school computer lab in Rio, Brazil. We exhibit an example of the development process of student's actions in an activity. Based on a macro-perspective, the analysis took into account a reconstruction of students argumentative strategies looking trough data focusing on the notions of anchorage and objectifying. Finally we discuss the validity of this model and how the social representation approach allowed us to have a global and a detailed local perspectives on students work.

## Introduction

The mathematical concept of function has been scrutinized in math education researches. According to Frege (1998) and Fischbein (1994), for someone who already grasped a concept, a representation would be a way of expressing this concept. It lead us to the discussion of a separation between mind and body and, external and internal world. A concept would pertain to the mind of an person while the representation would pertain to the external world. The relationship between concept and representation is found mainly in research findings about different representation for functions in a computer environment. (Herschkowitz and Schwartz 1997, Villareal and Borba 1998), Varella (1991), Campbell and Dawson (1995) are already against the dichotomy between external/internal world or subject/object based on the embodiment theory.

There is an alternative theory, the social representation, that also did not separate the internal/external world to investigate quotidian knowledge in a modern and complex society.

We try to answer the question - Is it possible to consider the processes of anchorage and objectifying, as proposed by social representation theory, in order to describe meaning production for functions?

## Social Representation and Argumentation: A fruitful relationship

Social representation theory has been developed since the 60's and takes in account that we face quick changes in our lives and to make sense of this modern and complex society forces us to build representations that help in dealing with the constant new knowledge. According to Jodelet (1984), *a social representation is a kind of knowledge elaborated and shared socially with an utilitarian goal, and provide grounding for building a common reality to a social group.* Moscovici (1978) emphasized two basic mechanisms of the process of building social representations, anchorage and objectifying. Anchorage is the process of integrating new ideas/concepts to an existed symbolic reference system and the changes implicit in that process. He adds, to anchor is to classify and nominate things that are not already nominated. We contributed to his definition of objectifying, for us to objectify is to summarize an idea or a set of ideas using images, schemes or metaphors.

Moscovici (1978) points out that there are two different kinds of social universe: the reified—characterized by scientific thought and the consensual—characterized by intellectual activities produced by daily social interactions; and in modern societies there is a strong relationship between these two universes. Almost all the time, new concepts are developed by scientists and they are appropriated by members of society. There is a need to turn new and not known concepts in familiar ones, quite constantly.

We classify mathematics classroom as a consensual universe because the purpose of learning scientific concepts in school is to enable students to use those concepts in their social life. Moreover, we assume that the process of producing mathematical concepts is similar to the process of producing social representations. When looking to students production of knowledge in classrooms, our focus is on their use of arguments to sustain or refute ideas.

According to Billig (1993), most of quotidian thoughts are established by argumentation. Thinking involves processes of accepting and rejecting, criticizing and justifying. A rhetorical approach to an object implies in a possibility of refuting an argument. His main concern related to social representation theory is that researchers did not value the process involved in argumentation in order to understand social thought.

We believe that argumentation theory (Perelman, 1996) is a strong tool to analyze social representations because during the anchorage process an object stop being the *thing that is talked about* to be the *thing we talk through*.

In the process of learning function, it is crucial to identify social representations produced by the students to this concept. It is the diversity of representations that are shared among the groups that will bring the concept to life. Usually, there are different points of view and, a group or a person has to convince others of the validity of their assumptions. Being that, it is in the process of argumentation that representations are built and changed, and at the same time representations provide substance for the arguments.

## Methodology

Our aim in this investigation was to better understand how students develop their conceptions about functions, their processes of meaning production, and to describe them from a theoretical point of view that combines social representation and argumentation theories.

The study took place in Rio, Brazil and the subjects were five 8th grade students who had not study function in school before. They met once a week, during a 90 minute periods for 2 months in a computer lab. The planning and organization of lessons aimed to privilege interaction among students and their speech. The *corpus* of analysis was compounded by students' speeches, their writing material and their work on the computer.

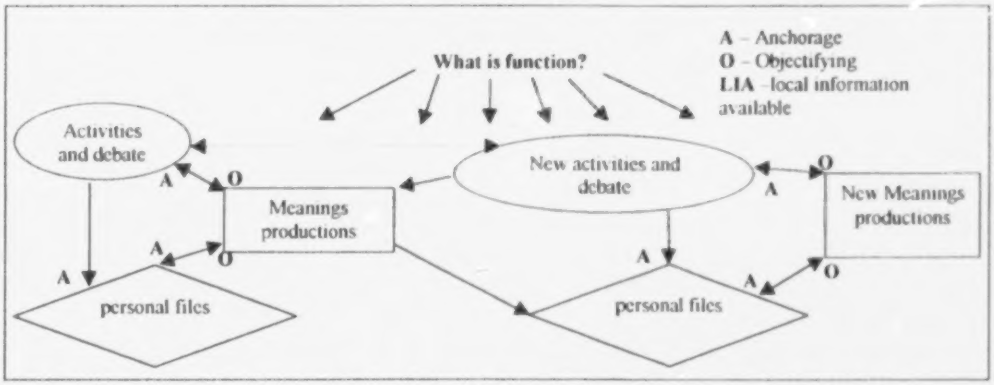
Students worked in small groups, and brought their findings to discuss with the whole group and the teacher. Two software were used, Greenglobes from Sunburst and TRM developed by Baruch Schwartz (1987). Every meeting, students were asked to write and talk about what they were thinking about what a function is. This method allowed us to identify how new meanings were being produced and what representations—content and form—were been used in the anchorage and objectifying processes.

Analysis focused on: 1) Rebuilding coherent sequences of reasoning; 2) Feeling in implicit spaces found in students' speech, 3) Identifying meaning productions; 4) Characterizing arguments through schemes; 5) Interpreting those schemes.

**Results**

Results from the analysis are briefly presented here.

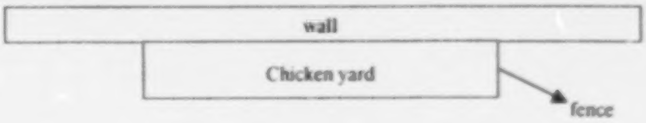
From the audio-tapes, written material and teacher's note, we found that meanings produced for function were, almost always, related to the activity the students were working in the day. The students never learned about function before so they did not have books or notes to help them answering "What a function is". The sources they had during this investigation were the proposed activities and the whole group debate. Besides those sources, each of them had what we called "personal files" – a set of experiences, knowledge, abilities that were built during each one life. In the below figure we illustrate the anchorage and objectifying processes, the meaning production process.


A – Anchorage
O – Objectifying
LIA – local information available
What is function?
Activities and debate
New activities and debate
Meanings productions
New Meanings productions
personal files
personal files
A
O
A
O
A
A
O
A
A
O

The main idea related to function by the students relied on graphs. Our option was to work on computers with the graphical representation for function opposed to the traditional emphasis on algebraic representation. Students' production met our expectation, because we promote an alternative field for producing meaning for function.

Activity 9

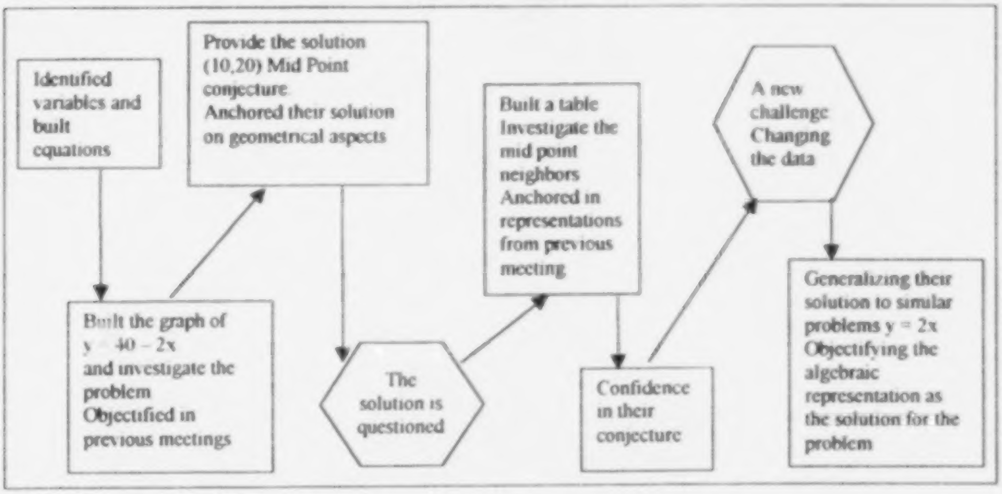
A farmer has 40 m of wire to build a rectangular fence for chickens. She will use a wall that is already built as a side of the chicken yard as shown in the below figure. Find the wide and length of this chicken yard in order to have the maximum surface possible.


wall
Chicken yard
fence

NOTE: In this activity, rather than given the right answer, I'm looking forward to see registered everything you thought and did while trying to solve the problem. You can use the TRM or Greenglobes to help you solving the problem.

The students started to solve the problem using a graphical representation approach and raised a conjecture that we categorized as "mid point conjecture".

Argumentative Strategy for the fence problem.


Identified variables and built equations
Built the graph of y = 40 – 2x and investigate the problem Objectified in previous meetings
Provide the solution (10,20) Mid Point conjecture. Anchored their solution on geometrical aspects
The solution is questioned
Built a table Investigate the mid point neighbors Anchored in representations from previous meeting
Confidence in their conjecture
A new challenge Changing the data
Generalizing their solution to similar problems y = 2x Objectifying the algebraic representation as the solution for the problem

The students named the wide and length of the rectangle by x and y. Then built the equations:

$S = xy$ $\quad 2p = 2x+y+m$ $\quad 2x+y=40$ $\quad 2p = m+40$

Then they used the TRM tc build the graph of $2x+y=40$. From that point on, they explored the graph using a possibility of this software to run the cursor over the graph. They realized that on the points where the graph intercepted the axes, (20,0) and (0,40), the surface was zero. They found mentally the surface for other values and conclude that at the point in the middle of the segment—the point (10,20)—the surface would reach the maximum value.

We questioned the students about their results, our expectation was that they would use a parable and not a line, or their algebraic representation. The students kept confident in their strategy. They modify it in one aspect, instead of looking to the graph to find the values for x and y to satisfy $y = 40-2x$, they used the table menu from the TRM.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| x | 9 | 40 | 8 | 7 | 15 | 9.5 | 9.9 | 9.999 | 11 | 10.5 |
| y | 22 | -40 | 24 | 26 | 10 | 21 | 20.2 | 20.002 | 18 | 19 |
| A | 198 | | 192 | 182 | 150 | 199.5 | 199.98 | 199.999998 | 198 | 199.5 |

Without abandoning their argument, students showed the validity of their conjecture, the closer to the mid point the maximum surface was obtained, and the maximum surface will be exactly at the mid point. To investigate their conjecture They tried numbers above and bellow 10, and observed what happened to the surface.

Their answer revealed a very original solution to this specific problem, moreover it reveals one important aspect—the students used the graphical representation as a tool for investigating the problem, not as an answer as it is found traditionally. But in order to validate their assumption they needed to use other representations such as algebraic and table.

## Conclusion

The results reified the role of technology in the teaching of function. Many of the meanings produced by the students happened because we were in a computer environment. The mid point solution might not appeared in a traditional classroom. It happened after their exploring TRM and negotiated different interpretations among them.

Regarding the articulation between social representation theory and meaning production process, we found that there is a possible marriage. Investigating meaning production for function based on the anchorage and objectifying process brought some advantages because it took in account the dynamic aspect of knowledge production.

It is worth to note that the model we used is only one among different ways of approaching meaning production for functions. Our option was mainly because we found many similarities between the construction of a social representation and the mathematics education phenomena that happened in schools. The results of this research are only a starting point to figure out at what extent this articulation is fertile. We hope that further research will follow.

## Bibliography

Bakhtin, Mikhail.1997. *Marxismo e Filosofia da Linguagem*, Hucitec, São Paulo, 8ª edição, p. 196.

Billig, Michael 1993. "Empirical approaches to social representations" in: *Studying the thinking society: social representations, rhetoric and attitudes*, Claredon Press, Oxford.

Campbell, S. & Dawson, A. J. 1995. "Learning as embodied action" in: *Exploiting mental imagery with computers in Mathematics Education, Springer*, p233-249.

Fischbein, E. 1994. *Intuition in science and Mathematics: an educational approach*, Kluwer Academic Publishers, 2a ed, Dordrecht.

Frege, G. 1998. *Lógica e Filosofia da Linguagem*, Editora Cultrix, São Paulo.

Hershkowitz, R. & Schwarz, B. 1997. "*Unifying cognitive and sociocultural aspects in research on learning the function concept*", in: proceedings of XXI PME, vol 1, p 148-164, Lahti, Finlândia.

Jodelet, D. 1984. "Représentations Sociales: phénomènes, concept et théorie." in: Moscovici, S. (ed) *Psychologie Sociale*, Presses Universitaires de France, Paris.

Moscovici, Serge. 1978. *A Representação Social da Psicanálise*, Zahar, Rio de Janeiro.

Nuñez, R. E., Edwards, L. D. & Matos, J. F. 1997. Embodied cognition as grounding for situatedness and context in Mathematics Education, Paper accepted for publication in: *Educational Studies in Mathematics*, July.

Perelman, C. & Olbrechts-Tyteca, L 1996. *Tratado da Argumentação - a nova retórica*, Martins Fontes, São Paulo, p 653.

Rabello de Castro, Monica. 1997. *Retóricas da rua: educador, criança e diálogos*, Editora Universitária Santa Úrsula/Amais Livraria e Editora, Rio de Janeiro

Rabello de Castro, M. & Frant, J. B. 1999. *A social representation approach to investigate learning*. Proceedings of PME 22, vol 1, Stellenbosch.

Schwarz, Baruch. 1987. The use of a *Microworld to improve ninth graders concept image of a function: the Triple Representations Model curriculum*, doctoral dissertation, Weizmann Institute, Jerusalém.

Schwarz, Baruch & Hershkowitz, Rina 1998. Prototypes: *Brakes or Levers in*

*Learning the Function Concept? The role of Computer Tools*, Paper submitted to the Journal for Research in Mathematics Education.

Varela, F. J., Thompson, E. & Rosch, E. 1991. *The embodied mind: Cognitive science and human experience*, MIT Press, Cambridge.

Villarreal, Monica E. & Borba, Marcelo C. 1998. – *Conceptions and graphical interpretations about derivative* – Proceedings of PME 22, vol 4, p-316, Stellenbosch.

# DEFINITION AS A TEACHING OBJECT: A PRELIMINARY STUDY

Fulvia Furinghetti* & Domingo Paola**
*Dipartimento di Matematica dell'Università di Genova. Italy
**Liceo Scientifico 'A. Issel', Finale Ligure (Sv). Italy

ABSTRACT. *In this paper we consider mathematical definition as a teaching object to be understood and used to perform mathematical tasks. There are studies on the way students define, but it seems to us that it is less considered how students perceive definitions, how they are able to work on them. In this paper we give a short account of ideas expressed by authors of the past and of the present on problematic features of definition. Afterwards we report an experiment with students aged 15 on defining trapezia. Since this experiment dealt with concepts not problematic for students it allowed to focus our study rather on the concept to be defined than on students' concept image of definition itself.*

## INTRODUCTION

In the activity of proving definition plays a very important role, nevertheless this role is sometime neglected. As pointed out by Marchini (1992) we can consider three different aspects of definition:
*Logical*: A definition can always be eliminated
*Epistemological*: A meaningful definition can not be eliminated
*Didactic*: When and how it has sense to introduce a definition. Which methods to use (examples, representations,...). Which aspects to emphasize (syntactic, semantic, ...). Which approach to adopt (constructive, declarative, ...). Which orientation to follow (historical, logical, ...).
As mathematics educators our work is mainly focused on the third aspect, but the first two aspects mentioned are also important in studying educational factors (both interpretation of students' behaviors and planning teaching units).

In this paper we present an activity in which classroom discussion was carried out around the definition of quadrilaterals. The focus of the described activity and of our analysis is not on the mathematical object to be defined (which, at least in principle, was well known by students), but on the definition itself considered as a mathematical object which needs of careful reflection.

## THEORETICAL FRAME

Some struggle about definitions is evidenced in textbooks and manuals. In many of them there is the misunderstanding which goes back to Euclid between undefined terms and terms to be defined. Often students' behavior is an 'echo' of this misunderstanding: they show difficulties in distinguishing between primitive 'terms' and terms which need to be defined, between definitions and descriptions, between sentences which are meaningful and circular sentences.

In the editions of Euclid commented by Commandino (in Latin and in Italian, Urbino, Italy, 1575) we find the well known statement 'A line is breadthless length' (Definition 2, Book I) accompanied by explanations taken from the physical reality. This habit of explaining in real terms what should be undefined lasted until recent times, introducing into mathematics a style similar to that of natural sciences. In a popular textbook of geometry for grade 6 (Asturaro-Burnengo, *Geometria*, Libro I per la prima classe della Scuola Media, 1941, unknown publisher) practical situations are evoked, see Fig.1, to explain the nature of the right line.

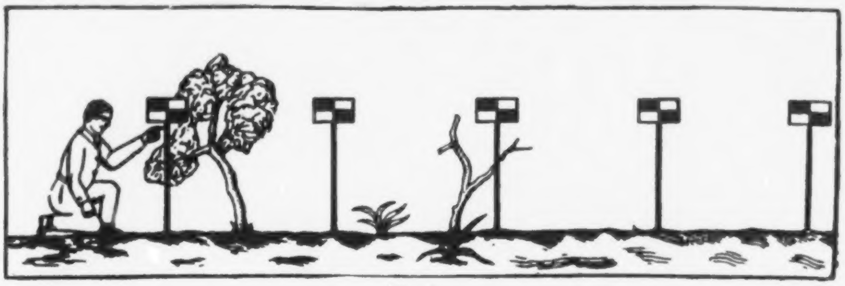


Fig.1

The French mathematician Adrien-Marie Legendre wrote a famous treatise of geometry which was published many times and was translated into foreign languages. In the editions in our library (Florence, 1818 in Italian, Paris, 1849 in French), we find this sentence: 'The straight line is the line having the minimum distance between two points'. This definition refers to an operational aspect and shows the influence of the empiricism in fashion at the Legendre's times. These examples may be considered as an evidence that to choose a certain presentation of the definition of a mathematical concept is linked with the view of mathematics and its teaching.

In the past mathematicians interested in mathematics education paid great attention to the problem of defining and wrote important works about it. The historian of mathematics David E. Smith (1911) emphasizes the didactic point of view, which for him means to stress the empirical side of definition, the distinction between 'basic' and 'merely informational' definitions. He takes from history (Thales, Euclid, Heron, Archimedes, Proclus, Apollonius, Leibniz, Gauss, ...) the materials for discussion. The logician Giuseppe Peano (1915; 1921) considers the didactic problem of definition from the logical point of view. The geometer and epistemologian Federigo Enriques (1927) discusses the same problem from the epistemological point of view. In his paper there is the interesting idea, which appears quite modern, that to define (which, among other things implies to classify, to generalize) may be seen as an unifying concept going across disciplines. As an example, he quotes the case of techniques to define used in chemistry. We point out that it is not surprising that Peano and Enriques paid such a great attention to the problem of defining, since both were important researchers in the field of foundation of mathematics when this kind of studies was particularly flourishing.

At the present time Marchini (1992) has developed didactic reflections on definitions and notations based on logic. He compares the Aristotle's and Euclid's approaches to definition. In the Aristotelian approach definition is not a cognitive tool, but is a tool to organize knowledge. The Euclidean approach also encompasses the 'genetic' introduction of definitions. Here the adjective 'genetic' is used to mean that the definition of an object specifies the meaning of the object by using only terms which have been already defined. Marchini (1992) points out that defining ecompasses aspects linked to proving, namely the inference rules and the choice of axioms of a theory. The role of definition depends on the view of mathematics adopted (realistic, formalistic, ...) and on the theory in which definition is set.

Pimm (1993) stresses the importance of studying the concept of definition from the educational point of view. In his review of the book (Borasi, 1991) he claims that the 'term definition is one of a handful of meta-mathematical marker terms (others include axiom, theorem, proof, lemma, proposition, corollary), terms which serve to indicate the purported status and function of various elements of written mathematics' (p.260-261). Afterwards he adds (p.261) 'How are we to come to learn about the functions these terms label, and the discriminatory power that can result from their use?'.

Important reflections on the activity of defining are raised by Harel and Tall (1991). From this article we take the following passage (p.39) which focus on the two basic issues - abstraction and generalization - intervening in defining:

> 'The process of formal definition in advanced mathematics actually consists of two distinct complementary processes. One is the *abstraction* of specific properties of one or more mathematical objects to form the basis of the definition of the new abstract mathematical object. The other is the process of *construction* of the abstract concept through logical deduction from the definition.
>
> The first of these processes we will call *formal abstraction*, in that it abstracts the form of the new concept through the selection of generative properties of one or more specific situations; [...]. This formal abstraction historically took many generations, but is now a preferred method of progress in building mathematical theories. The students rarely sees this part of the process. Instead (s)he is presented with the definition in terms of carefully selected properties as a *fait accompli*. When presented with the definition, the student is faced with the *naming* of the concept and the statement of a small number of properties or axioms. But the definition is more than a naming. It is the selection of generative properties suitable for deductive construction of the abstract concept'.

Previously the authors had distinguished three different kinds of generalization which depend on the individual's mental construction: *expansive*, *reconstructive*, *disjunctive*. Expansive generalization occurs when the student expands the range of applicability of an existing schema, reconstructive generalization occurs when the student reconstructs the existing schema, disjunctive generalization occurs when the student adds a new, disjoint, schema to those existing. Because of the difficulties involved in the construction process, Harel and Tall claim that the reconstruction occurring in defining activities is a *reconstructive generalization*.

Of course, the issue of definition concerns primarily teachers who have to be aware that 'In technical contexts, definitions might have extremely important roles. Not only that they help forming the concept image but they very often have a crucial role in cognitive tasks' (Vinner, 1991, p.69). Teachers also need to reflect that the way of presenting definitions is linked with their style in teaching mathematics. In defining many teachers are direct descendants of Euclid, since they consider definitions as something given *a priori* by an authority (the book, the teacher, ...) without any involvement of students. In this concern we found very interesting the use of results coming from etnomathematical research, as it was made by Gerdes (1988), to make teachers reflect on the role of axioms. The study which was used to this purpose concerns the alternative axioms applied by Mozambican peasants to build the rectangular basis of their houses. Analyzing these axioms makes teachers reflect on the role of axioms and pass from an authoritarian view of defining to a more flexible one based on construction of definitions.

We are convinced that the teacher has to give meaning to the activity of defining. De Villiers (1994) claims that students have to understand that certain definitions, certain procedures are chosen for their functionality at the interior of the mathematical domain. Definitions have to be functional to develop a theory. Thus he considers a third type of understanding (functional) in addition to the two types of understandings (relational, instrumental) discussed by Skemp (1976). In De Villiers's paper the discussion of the didactic use of definition is developed in the case of classification of quadrilaterals. The author has presented to students two types of classification:

- the hierarchical, e. g. 'Rectangles are parallelograms with four equal angles'
- the partitional, e. g. 'Rectangles are quadrilaterals with four equal angles and with adjacent sides of different length'.

In his experiment the author has found that students prefer partitional definitions. He makes the hypothesis that the language influences this preference: when it is said that a square 'is' a rectangle, to students the verb 'is' may evoke the identification of rectangles with squares and not the inclusion of the set of squares in the set of rectangles. The author recognizes the logical correctness of partitional definitions, but claims that from the didactic point of view hierarchical definitions are more efficient. He writes that to make students classify hierarchically it is useful to start from the partitional classifications and to let them encounter the disadvantages that this kind of definition has. We note that also in the experiment reported in (Fishbein, 1999) students in grade 9, 10, 11 show an orientation towards the partitional definition of quadrilaterals.

## DISCUSSION IN CLASSROOM ABOUT DEFINITION AND TRAPEZIA

In the following we report about an experiment in which we led students to reflect on definition through a classroom discussion on the ways of defining trapezia and on the consequences of the different definitions. The observer was the teacher himself who is also one of the authors of this paper (D. P.). The 19 students involved were aged 15. The school was an Italian Scientific Lyceum, which is a secondary school with a scientific orientation, including emphasis on mathematics. The time allowed for the activity has been two sessions. Students worked in groups formed according to their choice. The work in group was required because is functional to develop the discussion in classroom.

Students had some practice in the use of Cabri. In particular, they were acquainted with the central idea of this dynamic geometric software, namely that it preserves all the relationships set up among points, lines, and circles even when one of the basic components of the construction is dragged, that is, moved about the screen by the mouse.

The activity was part of our project aimed at introducing students to proof through the preliminaries stages of exploring and conjecturing. Our main goal was to make students aware of the importance of definition. The teacher preferred to work on defining a concept which was familiar to students in order their attention will be directed to the definition in itself rather than on the concept to be defined. In other cases we had experienced that to work on an unknown concept diverts students' attention from reflecting on definition as a mathematical object to the concept to be defined. We will see in the example presented that the discussion in classroom was orchestrated by the teacher who had in mind specific objectives. In our opinion discussion is a strong mediator to achieve cognitive tasks, since it puts in contrast different opinions and emphasizes the importance of side opinions (those which are held by a minority of students). Side opinions are important, since they act as a stimulus to consider different ways of reasoning. Teacher's interventions were mainly addressed to help students to state more and more exactly the definitions they had in mind. He fostered (by giving suitable hints) the production and discussion of examples and counterexamples in the various groups. When the discussion was slack the teacher suggested situations which could stress the low adequacy of the solutions proposed by students and push them to find new solutions.

The activity was developed in the way we report in the following. At the beginning the teacher reads the definition of trapezia given in the textbook ('qadrilaterals with two parallel sides and two non parallel sides'). From this definition students state that parallelograms are not trapezia. When they are asked to represent quadrilaterals with an Euler-Venn diagram they represent the set of parallelograms as a subset of the set of trapezia. We point out to the reader that their book of the preceding school grades contained this diagram, but the book presently used has a diagram consistent with the given definition. Students do not note the inconsistency in the two different representations of the definition of trapezia until the teacher stresses it. This behavior suggests that students consider definition as a mere description.

Students claim that trapezia can not be parallelograms since 'to be trapezia parallelograms must have two oblique sides'. One student notes that this is not true, since there are trapezia with one side perpendicular to the other two. At this point the teacher suggests to analyze the situation with Cabri. Students find that according to Cabri parallelograms are a special case of trapezia and thus they are disoriented and puzzled. For them trapezia are real objects (the one drawn in their textbook); their properties are fixed and described *a priori*, not generated by a definition.

At this point the teacher compares two different definitions of trapezia. One is that of students' textbook:

> 'Trapezia are quadrilaterals which have two parallel sides and two non parallel sides'.

Other authors, among them Enriques who has authored an important textbook of geometry for secondary school (translated into foreign languages), give this definition:

> 'Trapezia are quadrilaterals which have at least two parallel sides'.

The teacher raises the discussion by asking students to single out the advantages and disadvantages of the two definitions. Students note that the definition à la Enriques allows to make simpler Euler-Venn diagrams and it is in accordance with Cabri; nevertheless most students agree that the other definition is better since, as they put it, 'it is near to the our idea of the trapezium'. Only three students prefer the definition à la Enriques since it is 'more similar to the other definitions we deal with'. We guess that their feeling is originated by the fact that definitions they know are presented in the positive way, while the definition in which are mentioned 'two non parallel sides' contains a part presented in a negative way.

At this point the teacher stresses the necessity to be clear about the way definitions are generated and asks how isosceles trapezia can be defined. In the following we report literally the dialogue.

> Students: 'Isosceles trapezia have the two oblique sides congruent"
> Teacher: 'Does isosceles trapezia have symmetry axes?'
> Students (all): 'Yes'

To go on in the analysis of definition the teacher exploits the good occasion of discussion offered by those students who have chosen the definition à la Enriques and says them:

> 'According to your definition parallelograms are isosceles trapezia. But we know that they have not symmetry axes'.

Students are confused by this inconsistency and the teacher perceives that the situation is good to introduce the idea that definition have constraints, that is to say any choice in defining has consequences. In the case in question, since trapezia have been defined à la Enriques (at least two parallel sides), the statement 'Isosceles trapezia have symmetry axes' holds if isosceles trapezia are defined as 'trapezia with the angles adjacent to the base congruent'. Also one may define isosceles trapezia as 'trapezia which have symmetry axis'. With these definitions parallelograms are trapezia, but not isosceles trapezia. Moreover there are trapezia with the oblique sides congruent which are not isosceles trapezia, they are parallelograms. At this point the definition à la Enriques is definitely rejected by all students, since it appears that it implies to control too much elements.

It is interesting to note that, even if students have worked on symmetries before this experiment, only one student appreciated the role of symmetry in defining, the others used congruence.

## CONCLUSIONS AND HINTS FOR FUTURE DEVELOPMENTS

From the study here presented we may draw preliminary conclusions as well as hints for future developments which apply not only to the geometrical domain but also in other domains (algebra, analysis, etc.).

It seems to us true what Tall (1991) states, namely that defining is an issue marking the passage from elementary to advanced mathematical thinking. Activities such as abstraction, generalization, specification, establishing links among different representations are involved. This explains why students showed in defining the insecure behavior we have described. Our experiment is an empirical evidence of the fact that 'The abstract concept which satisfies *only* those properties that may be deduced from the definition *and no others* requires a massive reconstruction', see (Harel and Tall, 1991, p.39).

In the geometrical domain for students to classify means to distinguish objects which have different names. Thus they classify according to the names figures have, not according to figures' properties. When we add properties we perform an operation of 'specification' and restrict the set of figures having a given property. In making the union of sets of figures we perform an operation of generalization. Students pay more attention to the shape of figures than to properties. For the expert figures are defined and characterized by their properties; for the student figures exist and afterwards their properties are studied. Properties are adjectives added to figures.

The students' behavior in defining is similar to the behavior we found in studies on proof: they can check properties on single examples (= figures), but they have difficulties in generalizing (= grouping figures holding the same properties). The problem is that figures are perceived as something to draw, but behind them there is not any meaning, nor a constructive process which summarizes properties. Working on the consistency of definitions they already know helps students to give meaning to the drawing of figures. To bridge the gap between figures and their properties we plan to continue our activity on definition using the dynamic software Cabri more intensively than we did in the experiment here described. In our opinion working with Cabri in an appropriate way makes students reflect on the adequacy of the definitions they use to the microworld.

Eventually we point out that our experiment has shown that defining may be an activity which has not only a didactic value *per se*, but is also valuable as preparation to proof. As observed by De Villiers (1998), defining allows to develop understanding of the logical structure of 'if-then' statements. Defining also encompasses activities of discovering, in which reasoning has an abductive character, see (Cifarelli & Sáenz-Ludlow, 1996). In the case of the experiment we have described the work carried out by our students is similar to the work on open problems, as discussed in (Arsac, Germain & Mante, 1988), since:

- the statement of the problem is short ('Define a given mathematical object')
- the statement does not suggest the method of solution, nor the solution itself. There are not intermediate questions, nor requests of the type 'prove that ...'.
- the problem is set in a conceptual domain which students are familiar with (they are asked to define concepts that they are already using). Thus students are able to master the situation rather quickly and to get involved in attempts of conjecturing, planning solution paths and finding counterexamples. This happens, for example, when students have to control that all objects they are considering and only them are contemplated in a given definition.

For all these reasons the mathematical activity performed in classroom on definitions may result rich of meaning for students, since they are led to think that through defining are contributing to build parts of a mathematical theory.

## REFERENCES

Arsac, G., Germain, G. & Mante, M.: 1988, *Problème ouvert et situation – problème*, IREM, Villeurbanne.

Borasi, R.: 1991, *Learning mathematics through inquiry*, Heinemann Educational Books, Portsmouth, NH.

Cifarelli, V. & Sáenz-Ludlow, A.: 1998, 'Abductive processes and mathematics learning', in E. Jakubowski, D. Watkins & H. Biske (editors), *Proceedings of the Eighteenth annual meeting of the North American chapter of the international group PME*, 161-166.

de Villiers, M.: 1994, 'The role and function of a hierarchical classification of quadrilaterals', *For the learning of mathematics*, v.14, n.1, 11-18.

de Villiers, M.: 1998, 'To teach definitions in geometry or to teach to define?', in A. Olivier & K. Newstead (editors), *Proceedings of PME 22*, v.2, 248-255.

Enriques, F.: 1927, 'La definizione come problema scientifico', *Periodico di matematiche*, s.IV, v.VII, 72-82.

Fishbein, E.: 1999, 'Intuition and schemata in mathematical reasoning', *Educational studies in mathematics*, v.38, 11-50.

Gerdes, P.: 1988, 'On culture, geometrical thinking and mathematics education', *Educational studies in mathematics*, v.19, 137-162.

Harel, G. & Tall, D.: 1991, 'The general, the abstract, and the generic in advanced mathematics', *For the learning of mathematics*, v.11, n.1, 38-42.

Marchini, C.: 1992, 'Le definizioni e le notazioni: un problema didattico', *Quaderni Dipartimento di matematica dell'Università di Lecce*, Q.1, 125-143.

Peano, G.: 1915, 'Le definizioni per astrazione', *Bollettino della Mathesis*, a.VII, 106-120.

Peano, G.: 1921, 'Le definizioni in matematica', *Periodico di matematiche*, s.IV, v.I, 175-189.

Pimm, D.: 1993, 'Just a matter of definition', *Educational studies in mathematics*, v.25, 261-267.

Skemp, R. R.: 1976, 'Relational understanding and instrumental understanding', *Mathematics teaching*, v.77, 20-27.

Smith, D. E.: 1911, *The teaching of geometry*, Ginn and company, Boston.

Tall, D. (editor): 1991, *Advanced mathematical thinking*, Kluwer, Dordrecht.

Vinner, S.: 1991, 'The role of definitions in the teaching and learning of mathematics', in D. Tall (editor), *Advanced mathematical thinking*, Kluwer, Dordrecht, 65-81.

# When A Learning Situation Becomes A Problematic Learning Situation: The Case of Diagonals in the Quadrangle

Hagar Gal - The Hebrew University of Jerusalem, Yellin Teachers College, Israel
Liora Linchevski - The Hebrew University of Jerusalem, Israel

*Abstract*

*This paper discusses situations in which the teacher is unsuccessful in helping students overcome difficulties encountered during studying. We will identify a number of components of this situation, where the common factor is the teacher's lack of awareness of the student's thinking process. We will illustrate the identification and analysis of problematic learning situations involving the subject of the properties of diagonals in a quadrangle. We encountered these situations in the course of observing the teaching of geometry in junior high school intermediate and weak classes over an extended period of time. These and other problematic learning situations have provided the basis for constructing an intervention program for the training of mathematics teachers.*

## About The Research

Identifiable situations arise in the course of teaching where teachers have difficulties helping students that encounter problems in the learning process. We will call these situations *"problematic learning situations"* (PLS). A pilot study (Gal, 1998b) shows that such situations can be divided into two types:

1. Situations where the teacher does not identify that a student has a difficulty. This situation may result from the teacher's lack of awareness of possible difficulties. It may also result from the teacher's lack of mathematical knowledge.
2. Situations where the teacher identifies a problem but cannot produce an effective didactic response. This may be the result of insufficient pedagogical content knowledge, or failure to apply existing knowledge to particular difficulties that the student is encountering.

We observed over an extended period of time student teachers that taught geometry to junior high school students. The focus was on locating, identifying, and analyzing PLS from both - the student's and teacher's point of view. In this paper, we will illustrate PLS which we have identified in the area of diagonals in quadrangles, and discuss possible explanations for difficulties encountered.

## Theoretical Background

*Recognizing difficulties*. In this paper we refer to three categories of difficulties as reported in the literature. a) Students' difficulties in geometry (e.g. Close, 1982; Mitchelmore and White, 1996, Hershkowitz, 1989 and many others); b) Difficulties teachers have with geometrical concepts (e.g. Linchevski, 1985; Hershkowitz, 1989); and

c) Difficulties of teachers in coping with their students' difficulties in geometry (e.g. Gal and Vinner, 1997; Gal, 1998a).

*Analysis of difficulties.* The following theories and studies contribute to understanding problematic situations relating to diagonals in the quadrangle:

- Theories about visual perception (Anderson, 1995) can explain some of the difficulties relating to perception of shapes (e.g. the principles of Gestalt).
- Relevant theories about understanding mathematical concepts, prototypes, conceptual as opposed to pseudo-conceptual processes, and analytical as opposed to pseudo-analytical processes (e.g. Rosch and Mervis 1975, Vinner 1991, Vinner 1997).
- Van Hiele's theory examines the development of geometrical thinking and presents a five-level model: 1. Recognition 2. Analysis 3. Order 4. Deduction 5. Rigor. (For explanations of the levels see e.g. Hoffer, 1983.)

## Methodology and Findings

The research was carried out using a qualitative approach. Our research population included 12 student teachers in their third or fourth year of studies (of a four-year program), studying towards a B.Ed degree in teaching mathematics at junior high schools. These students (henceforth referred to as "teachers") taught their class for two hours a week in the course of the year, and acted as the class teacher in all respects. Data collection was carried out by means of interviews (semi-open) held before and after the class, observations, videotaping classes, and analyzing the teachers' lesson plans. Particular attention was paid to occurrences of students' misunderstanding or incorrect understanding and how the teacher dealt with it.

In this paper we demonstrate PLS concerning the diagonal, a concept usually taught at the junior high school level in the course of studying quadrangles. The following properties of diagonals are examined: 1. Equality. 2. Bisecting each other or the first bisects the second, while the second does not bisect the first. 3. Bisecting quadrangle's angles. (PLS relating to perpendicularity have been reported in Gal and Vinner, 1997).

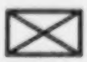
## Examples of difficulties observed and possible explanations

1) "Package" of properties. Many students understand the properties of diagonals of a polygon as a "package." In particular, the properties of equality of the diagonals and mutual bisecting are perceived as being bound up with each other.

*Example 1*

This case was observed in a 9th grade class (slow learners), where students had to work out which properties exist in the rectangle.

> Teacher (after identifying a problem with another student): *Elisha, what are bisecting diagonals?*
> Student 1 (Elisha): *They bisect each other and they're equal.* (Points to the diagonals) *First they bisect each other and after they bisect they're equal in their length.*
> Teacher: *I say that here the diagonals are equal. I've checked. (Shows using transparency) Now I say that they cross. What does that tell me?*



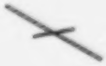
Student 1: *That they bisect at equal angles...*

Teacher: *You've marked that the diagonals bisect each other. You have to tell me why.*

Student 1: *Because they bisect each other. After they bisect, they're equal and they create right angles. These are right angles* (points to the vertex angles)

Teacher: *I'm not talking about the angles, I'm talking about the length of the sides, of these diagonals* (points)... *Draw me two straight lines that don't bisect.*

Student 1: (Draws)



Teacher: *They don't bisect?*

Student 1: *They don't bisect! Because they're not equal!*

Thus, in this example, the property of the diagonals as bisecting each other is related to additional properties: the equality of the length of the diagonals (and equality of vertex angles between them).

We will now try to understand the sources of this phenomenon.
Our experience shows that most dealings with quadrangles at junior high schools (as well as at elementary and preschools) relate to the most typical members of the category (Rosch and Mervis, 1975): the square, the rectangle, the parallelogram, and the rhombus. In addition, these quadrangles are generally dealt with by the most popular examples - the *prototype* (Hershkowitz, 1987). In these examples, the angles are not "too acute", and the differences in side lengths are not "too big". This situation makes it difficult to visually "infer" that the diagonals in a parallelogram or a rhombus are not equal to each other, or that in a rectangle and a parallelogram the diagonals do not bisect the angles. Anderson (1995) provides a theoretical underpinning for this claim: It is harder to compare things of a similar size than things of markedly different size. The result is the creation of a prototype of diagonals that enjoy all the "ideal" properties. Hence a generalization is made about all diagonals.
Furthermore, in these cases the possibility of non-reciprocal bisection is not considered whatsoever! Only the "kite", which belongs to the "less typical" members and frequently studied separately, allows a distinction between reciprocal and non-reciprocal bisection.
(ii) The aforementioned state of affairs contributes greatly to the idealizing of the properties of diagonals. According to Hoffer (1983), the phenomenon of idealization describes a situation in which the child interprets a word or an object in what we would consider a restricted manner by transferring it to a special case. Under these circumstances a limited interpretation is given to the equality of diagonals, and transferred to the specific case in which they bisect each other, and sometimes also bisect the angles. Similarly, the child attaches a limited interpretation to the fact that one diagonal bisects the other by transferring it to the case of reciprocal bisection.
(iii) Analysis of the kite and trapezoid supports the "package of properties": in these quadrangles, the diagonals are not equal, nor do they bisect. In other words, we again have the simultaneous existence /non-existence of two properties.
(iv) The principles of gestalt relate to the organization of visual information that reaches our eyes. *The principle of good continuity* explains our tendency to identify lines with better continuity than lines with sharp bends (Anderson, 1995). This is the reason why it

is difficult to look at parts of sections. The principle also explains why equality of parts is referred to as equality between entire sections.
The teacher, generally considering diagonals properties to be trivial, may not be aware of these barriers and their source, and thus be unable to aid students to cope with the difficulties.

2) Developmental levels according to Van Hiele's theory. According to this theory, judgment on level 1 is global and visual, and does not relate to the components of form and their properties. At this level the observer does not distinguish between diagonals that are equal, which bisect, and so on. The principle of good continuity makes the situation even more complicated. Judgment on level 2 is visual, with limited analytical ability - normally judgment is made according to the prototype. Hence, if the properties of the diagonals are taken from the prototypes, there may be a problem differentiating between cases where one property exists and the other does not. Instruction that does not take into account the student's level may result in PLS.

Let us now return to Example 1 and examine how it turns to be a PLS. The teacher clearly identifies the difficulty. However, she fails to take into account the global visual perception of the diagonals (characteristic of level 1) and assumes that the difficulty lies solely in the definition of the concept: the meaning of "equal" as opposed to that of "bisect." She also fails to distinguish between reciprocal and non-reciprocal bisection. That is why her guidance is insufficient.

3) Pseudo-conceptual behavior

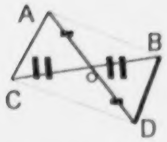
*Example 2* (continuation of Example 1)

Teacher: *What are "bisecting diagonals"?*
Student 2: *When they bisect each other.*
Teacher: *What does "bisect each other" mean?*
Student 2: *Don't know.*

The student's answer ("When they bisect each other") is a blatantly pseudo-conceptual answer (Vinner, 1997). The student uses the words which appear in the "name of the property" in order to describe the property. But these are mere words! No ideas whatsoever lie behind the words, as the teacher's questioning discloses. The student functioning on (van- Hiele) level 1 might perhaps cause this situation. Again, since the teacher is not aware of this possibility, the situation may evolve into PLS: This is a case of teacher's inability to characterize the source of the problem, let alone to find an answer to it as a result of the teacher's own insufficient cognitive knowledge. The teacher's insufficient knowledge of cognitive processes results not only in being unable to solve the problem, but also to characterize its source.

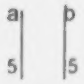
*Example 3*

This case was observed in a 9th grade class (average learners) on properties of the square.
Students were asked to answer the following question:



*Given that AO  OD and CO  OB, is ABCD a square?*
*If so, give proof of this. If not, provide an opposing example.*

The following difficulties arose. For example, one girl "superimposes" all four triangles according to "side, side" "pseudo-theorem" and concludes that the sides of the quadrangle are equal. The teacher suggests dropping this example and drawing one where the condition of bisecting diagonals does exist but the other properties of the quadrangle do not. The student draws a square and adds its diagonals. Another student says that he does not understand. The teacher suggests drawing all kinds of "squares" with bisecting diagonals and checking whether these are really squares, since it is difficult to see in the given example (is this true?).
Two students draw a square. The teacher tells them to try using other diagonals, but these must intersect in the middle. In another attempt to overcome the difficulty, the teacher draws two equal sections on the board, writes 5 cm on each of them and says:
*These are the diagonals, and they bisect... Draw a quadrangle... these sections are its diagonals. Will you necessarily finish up with a square?*



Here we have a task that calls for sufficient conditions for square-"from property to shape". This is a task on Van Hiele level 3 (or even 4 – requiring formalization). The task does not appear suitable for the students' level: what the girl says shows that the use of the congruence theorems is pseudo-analytical in nature (Vinner, 1997). When the student is required to draw conclusions about the equality of the four sides, she concludes that all four triangles are congruent according to "side, side". This pseudo-analytical behavior may result from difficulties functioning on level 3. Indeed, when the teacher suggests to her to check possible situations of the diagonals, the student operates "from shape to property" (level 2) - first drawing a square, and not "from property to shape" (level 3) - drawing diagonals first. This problem affects other students as well.
Here we face another PLS; teacher's lack of pedagogical and maybe mathematical knowledge prevents proper identification of the difficulty. Consequently the solution - of drawing examples - is not appropriate: a) implementing a solution of this kind (drawing the quadrangle first) on level 2 will not solve the problem (since prototype would be taken as a starting point). Nor does the teacher instruct the students to first draw the diagonals. b) The students do not have any aids such as dynamic software or transparencies, which will make it possible to run a dynamic check on the possibilities. Moreover, it might be that the teacher is not aware of the mathematical rigor required when differentiating between the various conditions: it would seem that she herself is confused and is dealing with additional requirement of equal diagonals. (She draws a case of equal lengths of diagonals: 5, 5.) In addition, she claims that it is difficult to make use of the original example (where it is difficult to judge whether the diagonals are equal, but easy to see that this is not a square!). Is this an instance where the teacher is also relating bisecting diagonals to equal diagonals, although keeping these two conditions are not assuring a square? Is that why she chose an example of equal diagonals?

4) Numerical treatment. We argue that caution is needed over applying numerical treatment as a tool for exercises. This is because exercises and numerical problems often trigger pseudo-conceptual or pseudo-analytical answers. In a "numerical solution," there is a temptation to carry out arithmetical manipulations by arbitrarily throwing out numbers taken from a situation (Clements et al., 1996). Our research also identifies many examples of such manipulations, but shortage of space precludes us from citing them here. Clements argues that examples could be seen of students who had not internalized the spatial ideas, but instead solved problems by means of arithmetic using effective problem-solving strategies. The following example illustrates teachers' tendencies to use a "numerical solution":

*Example 4*

In a conversation held with a teacher after the lesson on "bisecting diagonals" (Example 1), the teacher was questioned about checking whether the students understand the property of "bisecting diagonals".

> Teacher: *...You could give them a quadrangle, give them the numerical size of the diagonals, if there's a diagonal, for example, 6 cm long and this one is 4 cm, and then ask: "If the diagonals bisect, what does this tell you about the diagonals? What can you say about the sections of the diagonals?" Something like that, and then they'll need to know that this is two and this is two, and this is three and this is three. That's how I check whether they have understood that the right parts are equal.*

The numerical check proposed by the teacher solicits a pseudo-analytical response which uses the common arithmetical manipulation of dividing by two (halving). As a result, this is many times not an effective diagnostic tool for checking on students' understanding.

*Example 5*

In a conversation held with the teacher after the lesson on "bisecting diagonals" (Examples 1, 4), when asked if the students understood the teacher replied: "I think so," and then "Not all of them, I didn't manage to put it over."

> Researcher: *What's the problem in understanding this?*
> Teacher: *What's the problem? I don't know. I'm not sure that they understood.*

Later on, the teacher raises an anticipated possible difficulty: in order to check whether the diagonals bisect, the students will compare (equal) parts of different diagonals! But in fact, she does not think that there is any problem whatsoever here:

> Researcher: *So what's the problem?*
> Teacher: *What's the problem? No, there isn't a problem, if you ask me.*

This is an example, therefore, of a situation where the teacher fails entirely to recall students' difficulties, because of lack of pedagogical content knowledge. It should be noted that during the lesson, the teacher was aware to the difficulties that were revealed (see Example 1), but clearly any memory of this situation has been "erased".

In brief, our observation identified even more instances of basic difficulties with understanding the properties of diagonals:

- Bisecting diagonals (in a rectangle) means their bisecting the rectangle's angles.
- Equality of the diagonals means equality of the angles created at point of intersection
- Bisection as equality of wrong parts of the diagonals (one half-diagonal is equal to the half of the other diagonal), and in an additional instance as equality of sides.

## Summary And Conclusions

In this paper we have discussed and demonstrated PLS – problematic learning situations in which the student encounters difficulties and the teacher experiences problems when attempting to aid the student. The teacher may sometimes identify the existence of a difficulty, but fails to fully grasp how extensive it is, and assume it is the result of confusion, forgetfulness or related learning problems. The teacher may therefore choose to provide a reminder, repeat an explanation, or simply indicate the correct answer. The student may indicate that s/he has understood, since s/he may not be aware that s/he has not really understood. In many cases the student "co-operates" with the teacher by providing pseudo-conceptual or pseudo-analytical replies. On the surface, the situation appears to have been solved. However, seasoned teachers will explore the problem in greater depth, many times discovering that there is no understanding behind the answer. This is the crucial moment: does the teacher have the pedagogical knowledge, the didactic knowledge, and the mathematical knowledge to assist the student?

The following model describes this process. In figure 1, the teacher assumes that the student's knowledge and understanding is reflected as A. S/he maps out a learning process (shown by the arrow) which will lead the student's understanding to point B. But what if the student's understanding is really at point A' instead of A? If the teacher adopts the original intervention plan the student will arrive at position B', contrary to the teacher's plan (Fig. 2). The most appropriate response is to lead the student from point A' to A (shown by the double arrow in Fig. 3) or alternatively, to introduce a different learning process which will lead him/her from point A' to B (triple arrow). This model demonstrates the necessity of correctly identifying the student's understanding, exploring his/her thinking processes, and checking his/her understanding of the concept.

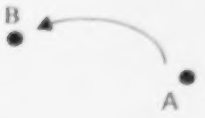


Figure 1

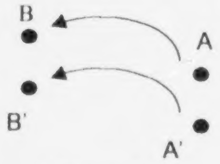


Figure 2

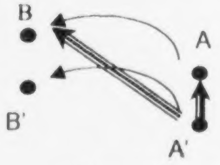


Figure 3

In this paper we presented several learning situations where the major difficulty of the teachers was the identification of the precis nature of students' difficulties (identifying situation A'). It was also showed that as a result the instruction process adapted accordingly, in order to ultimately bring the student to the desired point (B) failed. Not only did lesson plans fail to address the problems, there was no indication of anticipating difficulties of this kind at all. At times we also observed certain situations where teacher's own knowledge was not strong enough.

Therefore, pre-service teacher training should include a range of tools, such as mathematical knowledge, general and specific developmental theories, and treatment of misconceptions. However, these tools on their own are probably not enough for tackling PLS. The preparation program should also emphasize awareness of students' thinking processes, knowledge of cognitive processes, and mainly the ability to use this knowledge to plan in advance or to respond during classroom instruction to students' needs. However, details on a program of this type suggested by us would be beyond the scope of this paper.

## Bibliography

- Anderson, J. R. (1995). *Cognitive Psychology and Its Implications*. 4th ed. W. H. Freeman and Company, New York.
- Clements, D.H. Battista, M.T. Sarama, J. (1998). Development of Geometric and Measurement Ideas. In: Lehrer, L. Chazan, D. (Ed.) *Designing Learning Environments for Developing Understanding of Geometry and Space*. Laurence Erlbaum Associates, Publishers.
- Close, G.S. (1982). *Children's Understanding of Angle at Primary / Secondary Transfer Stage*. Polytechnic of the South Bank. London.
- Gal, H. (1998a). What Do They Really Think? What Students Think About the Median and Bisector of an Angle in the Triangle, What They Say and What Their Teachers Know About it. *Proceedings of the 22nd PME*, Stellenbosch, South Africa. 2, 321-328.
- Gal, H. (1998b). *Pre-service Teachers' Difficulties in Geometry Instruction. The case of the Angle.* Mofet Institute, Tel-Aviv. (Hebrew)
- Gal, H. Vinner, S. (1997). Perpendicular Lines - What Is the Problem? Pre-Service Teachers' Lack of Knowledge on How to Cope with Students' Difficulties. *Proceedings of PME 21*, Lahti, Finland. 2, 281-288.
- Hershkowitz, R. (1987). The Acquisition of Concepts and Misconcepts in Basic Geometry- or When "A Little Learning is A Dangerous Thing". In: Novak, J.D. (Ed.) *Proceedings of the Second International Seminar: Misconceptions and Educational Strategies in Science and Mathematics*. Vol. 3
- Hershkowitz, R. (1989). *Geometrical Concept Images of Students and Teachers*. Thesis Submitted for the Degree "Doctor of Philosophy". Hebrew University, Jerusalem. (Hebrew)
- Hoffer, A. (1983). Van Hiele-Based Research. In: Lesh, R. & Landau, M. (Ed.), *Acquisition of Mathematics Concepts and Processes*. Chap 7. N.Y. Academic Press.
- Linchevski, L. (1985). *An attempt at explication of the meaning of mathematical concepts used by primary school teachers*, Thesis Submitted for the Degree "Doctor of Philosophy". Hebrew University, Jerusalem
- Mitchelmore, M. C., & White, P. (1996). Children's Concepts of Turning: Dynamic or Static? *Proceedings of PME 20*. Valencia, Spain. 3, 415-421.
- Rosch, E. Mervis, C.B. (1975). Family Resemblances: Studies in the Internal Structure of Categories. In: *Cognitive Psychology* 7, 573-605. Academic Press, Inc.
- Vinner, S. (1991). The Role of Definitions in the Teaching and Learning of Mathematics. In: Tall, D. (1991). *Advanced Mathematical Thinking*. Kluwer Academic Publishers.
- Vinner, S. (1997). From Intuition to Inhibition - Mathematics, Education and other Endangered Species. *Proceedings of PME21*, Lahti, Finland. 1,63-78.

# From traditional blackboards to interactive whiteboards: a pilot study to inform system design

Christian Greiffenhagen, University of Oxford

*Interactive whiteboards are a new technology that has gradually found its way into classrooms. The aim of this study is to explore the potential of interactive whiteboards for the teaching and learning of mathematics. From field observations, videorecordings and interviews with a teacher this research develops a description of the teacher's use of a traditional board, and discusses how the teacher perceives the potential of an interactive whiteboard.*

## Rationale

The study reported here aims to explore the potential of interactive whiteboards for the teaching and learning of mathematics. Studying the use of a traditional board by a mathematics teacher in a secondary school raises two research questions:

1. In what ways did the teacher use the board?
2. How did the teacher perceive the teaching potential of an interactive whiteboard?

These two questions will be addressed after a brief outline of existing studies of educational technology for classrooms.

## Background

Research on Information and Communication Technology (ICT) in educational settings has been conducted mainly in two areas: (i) individual and pair learning software and (ii) distance learning (e.g., Kaput and Thompson, 1994). Rarely has there been research into technology for teaching in the 'traditional' whole-class classroom. Interactive whiteboards are a technology for the whole-class context that potentially offer a new way 'into the computer'. This study is guided by two broader questions:

- What *capacities* can this new technology offer for mathematics teaching?
- What are the *needs* for teaching and learning mathematics, which this technology might support?

These questions are approached from the perspective of *Requirements Engineering*, a branch of computer science that aims to determine what properties a system should have in order to succeed.

### *Requirements Engineering*

The introduction of new (computer) technology has not often been as successful as hoped (e.g., Selwyn, 1999). In search of a rationale for this failure, over the last decade researchers in the field of Computer-Supported Cooperative Work (CSCW) started to approach the design of technology from a new direction. They shifted focus from the individual person to the social setting, and from an idealized picture of work practices to the details and 'messiness' of everyday work practices. These 'workplace studies' (Button, 1993) were influenced by methods drawn from ethnography and anthropology, in particular ethnomethodologically informed interaction analysis (Garfinkel, 1967; Goodwin, 1981). Further, they usually make extensive use of repeated analysis of videotapes to address the everyday practices of participants that are easily overlooked (Hindmarsh and Heath, 1998).

This study aims to elicit requirements for a new software application for an interactive whiteboard that could be used in the mathematics classroom (Greiffenhagen, 1999).

### *Boards in mathematics teaching*

Many different types of boards can be found in schools: chalkboards, blackboards, whiteboards, markerboards, rollerboards, etc. However, there is a noticeable absence of research on the use of boards in teaching. They are usually simply regarded as large public displays. The few studies that have explored the advantages and disadvantages of boards come from a technological perspective. These studies (e.g., Stefik et al., 1987; O'Hare, 1993; Mynatt et al., 1999) identified the following common problems for the use of traditional boards in offices: (i) finding usable space among content that users did not want to erase; (ii) difficulty of sharing information following a discussion; and (iii) material once erased could not be recovered.

Interactive whiteboards provide the facility to modify the display electronically, and to save and print the displayed information. They can also be utilized in conjunction with video-conferencing. For mathematics, they offer the possibility of combining written text, symbols and diagrams in an electronic medium – which is hard to achieve using traditional computers with keyboards as the only input method.

Although they have been installed in many educational settings, such as the Classroom 2000 project[1], the NIMIS project[2] and the Collaborative Classroom project[3], key questions are rarely addressed: What could be their *educational* benefit for the teaching and learning of mathematics? What new facilities are offered for the mathematics classroom that could not be achieved with existing tools like textbooks or overhead projectors? This study aims to start addressing these questions.

## Methodology

The lessons of a single secondary school mathematics teacher were observed and video-recorded over a period of three months. Being a passive observer I would sit at the back of the classroom trying to interact as little as possible with the students. After each lesson, short semi-structured interviews were conducted, which focused on the observed lesson and opportunities for an interactive whiteboard. The shared experience of the observed lesson initiated the interview (Cooper and McIntyre, 1996) and grounded the discussion of how an interactive whiteboard might be used in events that actually happened in the lesson.

---

[1] http://www.cc.gatech.edu/fce/c2000/ (last updated August 1999)

2 http://collide.informatik.uni-duisburg.de/Projects/nimis/ (last updated June 1999)

3 http://dcr.rpi.edu/ (last updated December 1997)

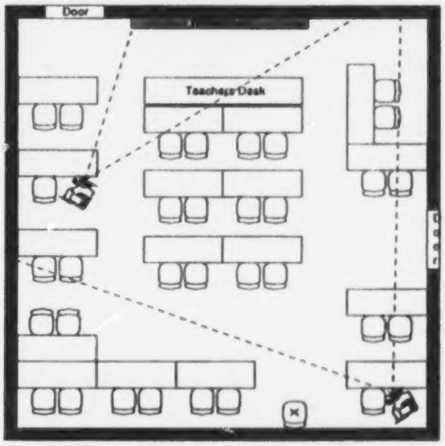


**Figure 1: Placement of cameras**

To gain insights into the everyday practices of using the board, the lessons were recorded on video to form the basis of an 'interaction analysis' at a later stage. Two cameras were placed in the classroom. The first was aimed directly at the board to capture local interactions. The second camera, in one of the rear corners, aimed to capture as much as possible of the general classroom interaction (see Figure 1). Thus, while still focusing on the teacher, the classroom could be observed as well. With a special machine it was possible to view both tapes simultaneously, focusing on the items written on the board as well as the interaction in the class.

One advantage was that once the cameras were set up, no one was needed to 'direct' them, and I could sit away from the cameras creating less disturbance. The main disadvantage was that only the back of the students' heads was recorded. This was due to the initial focus on the teacher rather than the students. Through the repeated analysis of the videotapes, it was possible to focus on the everyday actions performed at the board; actions which are otherwise hard to observe and analyze.

### 1. In what ways did this teacher use the board?

Usually, each observed lesson was divided into two parts. In the first part, the teacher introduced a new topic, or revised what had been learnt earlier, in a whole-class setting. In the second part, the students would work on their own or in pairs on textbook questions. The board was only used during the first part. This organization seems to be fairly typical in mathematics teaching, as Jaworski (1994, p.8) observes:

> Typically, the teacher introduced the mathematical content of a lesson using exposition and explanation (teacher talk), usually from the front of the classroom (using blackboard and chalk). Pupils were then given exercises through which they practiced the topics introduced by the teacher.

From a constructivist point of view (Jaworski, 1994), in order to learn mathematics, students need to be brought into contact with mathematical concepts in a way that allows sense-making and cognitive structuring. Creating a classroom discourse that raises and questions mathematical ideas would draw students into a mathematical world in which mathematical sense-making is an active part of communication, thus making it possible for individuals to access and process mathematical ideas. Hence, the question arises whether interactive whiteboards could be used to create an environment in which students are more actively involved in the lesson – for example, through the use of electronic tablets or radio-mice, which students could use to 'write' from their local vicinity, the desk.

The analysis of the videotapes identified several actions on the board: contact with the board; pointing by the teacher; and by the student – which will be described below:

*Contact with the board*

The teacher used various ways to erase items on the board, sometimes a proper eraser, but usually his hand or part of his shirt. These events could occur immediately one after the other. On one occasion, the teacher initially used an eraser, followed only seconds later by his hand. The board was touched to (i) write; (ii) erase; (iii) point; and (iv) balance while writing.

These could occur simultaneously. For example, a student was observed writing on the board with his right hand while leaning on it with his left arm. These different ways of touching the board hold implications for the physical ergonomics of electronic whiteboards. There are two different types of boards: touch-sensitive ones, and ones that are written upon with an electronic pen. They create different sets of problems: Using the touch-sensitive board the student would have created signals both with the pen and his arm. The signals from his arm would be interpreted as undesired 'writing' on the board. Using the second type of board, erasing would only be possible by using an electronic pen or wiper.

*Pointing by the teacher*

The teacher often pointed at the objects on the board. This was done to refer to what he was talking about, or to confirm what he had just said. For example, in one lesson the teacher started by drawing two axes. Having asked the students which one was the x-axis and which was the y-axis, he labeled them accordingly. He then continued:

> These are things you need to remember: x-axis is the horizontal axis ((hand moves from left to right)) and y-axis is the vertical axis ((hand moves from bottom to top)).

One way to think about this is in terms of *resources*. Generally, two kinds of resources are to be distinguished: transient and persistent ones. In the example above, a transient resource (pointing) was used to reinforce a persistent resource (the drawing on the board). Pointing by the teacher was also used to reinforce a reference made by the student[4]:

> The teacher has drawn two axes on the board; writes "y=x+1" and "y=x+2".
> T: What do their graphs look like?
> P: The last number goes through the origin (.) that number.
> T: This number ((points at "1", leaves hand there))

---

[4] The notation found in the transcripts is derived from the conventions described in Goodwin (1981). '(.)' denotes a silence of a tenth of a second. A colon ':' indicates that the sound preceding is noticeably lengthened. A bracket '[' connecting the talk of different speakers shows that overlapping talk begins at that point. Comments are displayed in double parentheses: '((comment))'. The teacher is referred to as 'T' and pupils as 'P', 'P2', etc.

The student verbally referred to "the last number". The teacher then put his hand there, making it visible for the whole class and reinforcing what the student has just said.

Transient resources are available only at that particular point of time (in contrast to persistent resources). Therefore, students who do not pay attention when the teacher is talking and writing on the board lose what has been said (the transient resource) while still having access to what has been written (the persistent resource). However, what has been written might have an incomplete or different meaning without the additional verbal information. Hence, the meaning of the persistent resource (the writing on the board) might change over time through the availability of the transient resource (teacher's and student's talk).

The architecture of writing and drawing form an essential part of mathematics. "Mathematics is perceived overwhelmingly written." (Pimm, 1987, p.1) In fact, "*being thought* in mathematics always comes woven into and inseparable from *being written*" (Rotman, 1993, p.x). One of the problems of conventional computers is the restriction to typed test – in contrast to using pencil and paper, which allow a mixture of text, symbols and various forms of diagrams. There are few other subjects in which the writing on the board is so much the focus of attention during the lesson. This is one of the reasons why interactive whiteboards with their ability to record, highlight and save the content of the board might be beneficial for mathematics teaching. In addition, they might provide resources for students to point at the board:

### *Pointing by the student*

How can students point to objects on the board while staying in their seats? What difficulties do they have to overcome? An example will be used to illustrate this:

> The teacher has drawn two axes on the board and has just explained that there are four "quadrants". Students are James (J) and Rachel (R).

T: Which is the first quadrant? James? ((James sits in the first row))
J: ((points at second quadrant))
T: This one? ((Figure 2))
((points at second quadrant))
No, it's not this one.
[…]
Which one is it? Rachel?
((Rachel sits in the last row))
R: It's
[
T: ((points at first quadrant)
R: (.) It's ah:::
[
T: ((hesitates, pulls hand back))
R: It's ah (.) where they are all positive.
T: Where they are all positive. ((puts hand in first quadrant))

**Figure 2: Student and teacher pointing**

This example demonstrates the different resources available to students: The first student, James, was able to point at the board because he was sitting at the front, whereas the second student, Rachel, had to use words to describe the location – which is a very indexical way of transmitting information. As mentioned above, mathematicians find it very difficult to only talk about concepts without having a piece of paper or board to illustrate; they usually talk *and* write simultaneously. Interactive whiteboards with the support of electronic tablets for students would provide students with the opportunity to say *and* point to their suggested quadrant.

In fact, having electronic tablets available for every student would provide the means for asking new types of questions. Rather than asking James where the quadrant was, the teacher could have asked *every* student to point at their tablet where they thought the first quadrant was. These answers could then be used as a basis for discussion among the whole class. During an interview, the teacher came up with a similar idea:

> When all the students could have some tablet or something. […] I mean, that would be good. […] you could kind of, almost "what do you think the answer to this question is?", couldn't you? ((laughs)) You could have some software saying, "Well, ten of you think the answer is this, and six of you think the answer is that." And you could discuss the diff (.) You know why some people think that this is the answer and why others think that that is the answer. And you could get people who perhaps haven't got the correct answer to explain what they are doing. (Interview, 18.6.99, p.9)

Through such discussion, the mathematical concepts become part of the communicative discourse of the classroom through which individuals can start to build their own sense of them.

**2. How did the teacher perceive the teaching potential of an interactive whiteboard?**

One of the most obvious features of the interactive whiteboard is that it can be used to save the notes on the board, which can be printed at a later stage. Other features could be pre-prepared grids (e.g., 1mm, 3D, or isometric) or the possibility of annotating existing notes or other software applications. Because all the notes on the board are digital, the board could be used like a piece of paper, i.e., rotated or flipped (e.g., to demonstrate how to draw a particular graph). These facilities are specific to *mathematics* because they focus on particular aspects of writing. They have therefore not yet been implemented in the software provided with interactive whiteboards.

These features might all be 'useful' and facilitate the teaching, but they are only facilities for a better presentation by the teacher. They would not provide new resources for communication in the classroom:

> That's just like facilities which a board could have, isn't it? It's not really improving the quality of the mathematics, which is going on in the classroom, which would be really good, wouldn't it? If you could produce something which would improve kid's ability to

> communicate mathematics (.) otherwise you would just produce a glorified blackboard, aren't you? A sort of high-tech blackboard. (Interview, 18.6.99, p.11)

Focusing primarily on the teacher, it is easy to think of an interactive whiteboard as "a glorified blackboard". This has been the approach of most manufacturers and teachers. The main finding from this study however is the importance of focusing on the possibilities of enhancing the communication and interaction of the *students* to potentially achieve a real educational benefit.

As mentioned above, mathematics "is perceived overwhelmingly written" (Pimm, 1987). Focusing on students' writing both aims to teach the children about mathematics as well as written communication. Mehan (1989) investigating the use of computers observed that "having an audience [...] gave students a purpose for writing". Interactive whiteboards might present students with an *audience*, by providing the opportunity to display the work of students quickly on the board. Audiences could be the whole class or students in a different school.

At the moment, the teacher often tells students that they should not only write down the answer and that they should not forget their audience. But this advice is superfluous because the students are always aware of their audience – the teacher (cf. Morgan, 1988, p.45). This mathematics teacher remarked:

> Yeah, it's giving them, it's giving them audience, isn't it, for their work? Perhaps if they are just writing things in their exercise books, it's between you, them and perhaps their parents look at their exercise books occasionally, you know? [...] It's a very narrow audience, isn't it? In terms of who they are communicating to. (Interview, 18.6.99, p.10)

With the new technological opportunities, the work of students could be displayed quickly on the board, giving them instant feedback. It would also mean that more text written by students (rather than the teacher) might be displayed on the board:

> I mean, it's (.) when they (.) if they saw it in their own work, you know, in another student's work, it'll perhaps have more significance than when you do on the board. You can talk to them just about how to set things out properly and how to communicate things properly. Ahh, they kind of think "that's the way the teacher does it". But if they could see a good example of another student doing it. (Interview, 21.5.99, p.5)

In other words, seeing the answer of a fellow student on the board might help students to develop the mathematical concepts involved in their own language.

## Conclusion

This study has started to address some of the issues of interactive whiteboards in mathematics teaching. Traditionally, the use of interactive whiteboards has been restricted to presentations made by the teacher. In contrast, our two main findings are:

1. Interactive whiteboards should not only be seen as a *presentational* device for the teacher, but as an *interactive and communicative* device to enhance the communication by the students; and

2. Interactive whiteboards might provide new resources to focus on the *writing by students* in the mathematics classroom. In particular, they could potentially provide students with an audience and allow the teacher to display more work written by students on the board.

By pursuing these two points, it is hoped that interactive whiteboards will provide innovative resources for teaching and learning mathematics – in contrast to Krummheuer's observation (1992, p.214; my translation):

> The potential of computer technology is not being used to improve lessons but only to imitate them.

## References

Button, G. (Ed.) (1993). *Technology in Working Order*. London: Routledge.

Cooper, P. and D. McIntyre (1996). *Effective Teaching and Learning: Teachers' and Students' Perspectives*. Buckingham: Open University Press.

Garfinkel, H. (1967). *Studies in Ethnomethodology*. Englewood Cliffs: Prentice-Hall.

Goodwin, C. (1981). *Conversational Organization*. New York: Academic Press.

Greiffenhagen, C. (1999). The use of the board in the mathematics classroom: a pilot study to inform technology design. Master's thesis, Oxford University. Available as http://users.ox.ac.uk/~scat0800/Papers/msc_erm.ps

Hindmarsh, J. and C. Heath (1998). Video and the analysis of objects in action. *Communication & Cognition 31*(2-3), 111-130.

Jaworski, B. (1994). *Investigating Mathematics Teaching: A Constructivist Enquiry*. London: Falmer.

Kaput, J. J. and P. W. Thompson. Technology in mathematics education research. *Journal for Research in Mathematics Education 25*(6): 676-684.

Krummheuer, G. (1992). *Lernen mit "Format"*. Weinheim: Deutscher Studien Verlag.

Mehan, H. (1980). Microcomputers in classrooms: Educational technology or social practice? *Anthropology and Education Quarterly 20*(1): 4-22.

Morgan, C. (1998). *Writing Mathematically*. London: Falmer.

Mynatt, E.D., T. Igarashi, W. K. Edwards, and A. La Marca (1999). Flatland: New dimensions in office whiteboard. In *Proceedings of CHI'99*, pp. 346-353.

O'Hare, M. 1993). Talk and chalk: The blackboard as an intellectual tool. *Journal of Policy Analysis and Management 12(1)*. 238-246.

Pimm, D. (1987). *Speaking Mathematically*. London: Routledge.

Rotman, B. (1993). *Ad Infinitum ... The Ghost in Turing's Machine*. Stanford: Stanford University Press.

Selwyn, N. (1999). Why the computer is not dominating schools: A failure of policy or a failure of practice? *Cambridge Journal of Education 29*(1), 77-91.

Stefik, M., G. Foster, D. G. Bobrow, K. Kahn, S. Lanning, and L. Suchman (1987). Beyond the chalkboard. *Communications of the ACM 30*(1), 32-47.